Qualifying/Review Exam in Topology

5–18–90

Part I. State the following theorems:
(a) Tietze Extension Theorem (b) Brouwer Fixed Point Theorem

Part II. Let $f : X \to Y$ be a continuous surjection between topological spaces. Which of the following properties hold for Y whenever they hold for X? (Give a proof or a counterexample with justification.)
(a) compact (b) normal (c) connected

Part III. Choose four of the following six problems to work:

1. Let X be a Hausdorff space.
   (a) If $S \subset X$ is a finite set then there is a collection $\{U_s \mid s \in S\}$ of pairwise disjoint open sets in X for which $s \in U_s$.
   (b) If S is infinite then the conclusion of part (a) may or may not be true.

2. (a) Let X be a topological space and let $\sim$ be an equivalence relation on X. Describe the quotient topology on the set of equivalence classes $X/\sim$.
   (b) For $x, y \in R^2 - \{\mathbf{0}\}$ put $x \sim y$ if and only if x and y lie on a line through the origin. Show that $R^2 - \{\mathbf{0}\}/\sim$ with the quotient topology is homeomorphic to S^1.

3. Let (X, d) be a metric space and let $f : X \to X$ be a continuous function which has no fixed points.
   (a) If X is compact then there is an $\epsilon > 0$ such that $d(x, f(x)) > \epsilon$ for all $x \in X$. (Be sure to clearly identify any criteria for compactness which you use.)
   (b) Show that the result of (a) is false when compactness is not assumed.

4. Let $\mathcal{B}$ be the collection of lines in R^2 with slope equal to 1.
   (a) Show that $\mathcal{B}$ forms a basis for a topology τ on R^2.
   (b) Describe the subspace topology which τ induces on the x-axis.
   (c) Prove that (R^2, τ) is homeomorphic to $(R, \tau_d) \times (R, \tau_i)$ where τ_d denotes the discrete topology and τ_i denotes the indiscrete topology.

5. Let X be a topological space with subspaces A, B, D.
   (a) Suppose that D is dense in X. Is every element of X a limit point of D?
   (b) Let $A \subset B$. Show that if A is dense in B then it is dense in $\overline{B}$.
   (c) Show that if D is dense in X then $D \cap A$ need not be dense in A.
   (d) If the only dense subset of X is X itself what can be said about X?

6. For each $\alpha \in J$ let X_α be a topological space whose topology is τ_α, and which contains at least two elements.
   (a) Describe the product topology on $\prod_{\alpha \in J} X_\alpha$.
   (b) Show that each projection map $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ is an open map.
   (c) Establish necessary and sufficient conditions on the collection $\{(X_\alpha, \tau_\alpha)\}$ so that $\prod_{\alpha \in J} X_\alpha$ is discrete.

PART IV. Work two of the following three problems:

1. Let $f : (X, x) \to (Y, y)$ be continuous.
   (a) Describe $f_* : \pi(X, x) \to \pi(Y, y)$ and verify that it is well-defined.
   (b) Give examples showing that if f is surjective then f_* may or may not be surjective.
   (c) Give examples showing that if f is injective then f_* may or may not be injective.

2. (a) Define contractibility of a space X.
   (b) Show that if X is contractible then every continuous function $f : Y \to X$ is homotopic to a constant function.
   (c) If X is a space with a contractible universal covering space and $f : (Y, y) \to (X, x)$ is a continuous function whose induced homomorphism $f_* : \pi(Y, y) \to \pi(X, x)$ is trivial then f is homotopic to a constant map.

3. (a) Define the term n-*manifold*.
   (b) Let M be an n-manifold where $n \geq 3$ and let x and y be distinct points in M. Show that $\pi(M, x) \cong \pi(M - y, x)$.

PART V. Work one of the following three problems:

1. (a) Show that a connected locally path-connected space is path-connected.
   (b) Show that in a locally path-connected space the path components are both open and closed.
   (c) If the path components of a space are open does that imply that the space is locally path-connected?

2. (a) A subspace of a locally compact Hausdorff space X is locally compact if and only if it is of the form $U \cap C$ where U is open and C is closed (in X).
   (b) Neither the rationals nor the irrationals is locally compact with the Euclidean topology.

3. Prove that neither $\overline{S}_\Omega$ nor S_Ω are metrizable, but both are regular. Also determine which of them are Lindelöf.