Algebra Qualifier

August 24, 1990

Instructions: Do as many of the problems as you can. Try at least one from each section: Group Theory, Ring Theory, Module Theory, Field Theory and one from the Elective section.

I. Group Theory

1. Let $G = AB$ be the semi-direct product of a group A by a group B (i.e. $B \lhd G,\ G = AB$ and $A \cap B = 1$). Let $[A, B]$ denote the subgroup of G generated by all commutators $[a, b] = a^{-1}b^{-1}ab$ where $a \in A,\ b \in B$.
   (a) Prove that $[A, B]$ is a normal subgroup of G contained in B;
   (b) Prove that $G/[A, B] \cong A \times (B/[A, B])$.

2 Let $\mathcal{S}$ be the set of Sylow 7-subgroups in a simple group G of order 168. If $S \in \mathcal{S}$, show that S acts transitively on $\mathcal{S} \setminus \{S\}$ by conjugation.

II. Ring Theory

3. Let p be a prime ideal in a commutative ring R with 1. Let $S = R \setminus p$.
   (a) Show S is a multiplicative subset of R
   (b) Show that $S^{-1}Rp$ is the unique maximal ideal in $S^{-1}R$.

4. Let $k[x, y]$ be the polynomial ring in two indeterminants x, y over a field k. Show that $k[x, y]$ is a unique factorization domain but $k[x, y]$ is not a principal ideal domain.

5. Let k be a field and $T_2(k)$ the ring of 2×2 upper triangular matrices.
   (a) Show that $T_2(k)$ contains a unique maximal nilpotent 2-sided ideal. Find it explicitly.
   (b) Formulate a generalization of (a) and try to prove it.

III. Module Theory

6. Let A be a finite dimensional algebra over a field k. If e is an idempotent in A show that the left A-module $(eA)^*$ is injective. (Here $(eA)^*$ is the dual space $Hom_k(eA, k)$ and eA is a right A-module where A acts by right multiplication.) *Hint:* Show first that the right A-module eA is projective.

7. Let k be an algebraically closed field of prime characteristic $p > 0$. Let G be a cyclic group of prime order q.
   (a) Show the group algebra $k[G]$ is isomorphic to $k[x]/(x^q - 1)$ where $k[x]$ is the polynomial ring in one variable. (Recall $k[G]$ is the ring whose elements are the linear combinations $\sum_{g \in G} a_g g$ with $a_g \in k$ and where multiplication is given by
$$\left(\sum_{g\in G} a_g g\right)\left(\sum_{g\in G} b_g g\right) = \sum_{h\in G}\left(\sum_{mn=h} a_m a_n\right) h\,.$$
   (b) Show that if $q \neq p$, then every finite dimensional $k[G]$-module is projective.

8. Let M be a left A-module for a ring A with 1. Show that the functor $- \otimes_A M$ from the category of right A-modules to the category of abelian groups is right exact. Give an example of a ring A and an exact sequence of right A-modules
$$0 \to U \to V \to W \to 0$$
and a left A-module M such that the sequence
$$0 \to U \otimes_A M \to V \otimes_A M \to W \otimes_A M \to 0$$
is *not* exact.

IV. Field Theory

9. (a) Give an example of a finite separable field extension E/k which is not Galois. Prove your answer is correct.
(b) Give an example of a finite purely inseparable field extension E/k. Prove your answer is correct.

10. Let $p > q$ be primes. Let G be a non-abelian group of order pq. If E/k is a Galois extension with group G discuss the subfields of E.

11. Let $k(x)$ be the quotient field of the polynomial ring in one indeterminant x. Prove that every element $r(x) = p(x)/q(x)$ which is not in k is transcendental over k. What can you say about the extension $k(x)/k(r)$?

V. Electives

12. Let k be an algebraically closed field of characteristic $p > 0$. Let V be a vector space over k whose dimension is at most 4. Let T be a linear transformation on V whose minimum polynomial divides $x^p - 1$. What are the possible Jordan canonical forms for T?

13. Suppose V is a finite dimensional vector space over the field $\mathbb{Q}$ of rational numbers. Let $T : V \to V$ be a $\mathbb{Q}$-linear transformation on V and assume T has minimum polynomial $(x-1)^3(x-2)^2$. Find subspaces V_1, V_2 satisfying the conditions:
    (a) $T(V_i) \subseteq V_i$;
    (b) The minimum polynomial of T on V_1 is $(x-1)^3$;
    (c) The minimum polynomial of T on V_2 is $(x-2)^2$.

    Your answer should express V_i in terms of T and V.