Qualifying/Review Exam in Topology

8–22–90

Instructions: There are eight problems labelled I through VIII. For each problem you should work just one part labelled as 1,2, etc. You may work additional parts if extra time is available.

I. 1. Let X be a topological space and let Y be a subset of X endowed with the subspace topology.

(a) If $\mathcal{B}$ be a basis for the topology on X, show that $\{U \cap Y \mid U \in \mathcal{B}\}$ is a basis for the subspace topology on Y.

(b) Which of the following four properties hold for Y whenever they hold for X? second countability; local connectedness; regularity; normality. (Prove or describe a counterexample.)

II. 1. (a) Show that a closed subset of a compact space is compact.

(b) Let $f : X \to Y$ be a continuous bijection where X is compact. If Y is a Hausdorff space then f is a homeomorphism.

(c) Does the result of (b) still hold if one only assumes that Y is a space in which all singletons are closed?

III. 1. A non-Euclidean topology known as the *Zariski topology* can be put on $\mathbf{R}^n$ by declaring a set C to be closed iff there is a set (possibly empty) of polynomials in n variables with real coefficients such that C is the set of points in $\mathbf{R}^n$ on which all of the polynomials vanish.

(a) Verify that this describes a topology on $\mathbf{R}^n$.

(b) For $n = 1$ show that the Zariski topology is a familiar topology on $\mathbf{R}$.

(c) Show that the Zariski topology on $\mathbf{R}^n$ is not Hausdorff. (hint: consider a coordinate axis.)

2. Let X be an uncountable set and $x_0 \in X$. A topology is given by declaring a set $C \subset X$ to be closed iff either C is countable or $x_0 \in C$.

(a) Verify that this describes a topology on X.

(b) The topology is T_2.

(c) The topology is not metrizable.

IV. 1. (a) Prove that a path connected space is connected.

(b) Let $X = \{a, b, c\}$ be given the topology $\tau = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$. How many path components and how many components does X have?

2. Let X be locally connected, and let $f : X \to Y$ be a closed continuous surjection. Show that Y is locally connected.

V. 1. Let X be a metric space, $A \subset X$ and $x \in X$. Show that x is in the closure of A if and only if there is a sequence (a_n) in A which converges to x.

VI. 1. (a) State Urysohn's Metrization Theorem.

(b) Let $f : X \to Y$ be a continuous function where X is a compact metric space and Y is a Hausdorff space. Show that $f(X)$ is metrizable.

2. (a) State Urysohn's Lemma.

(b) Let X be a connected normal space, and let A be a proper closed subset of X. Show that there is a continuous surjection $f : X \to S^1$ such that $f(a) = 1$ for all $a \in A$.

VII. 1. Let X be a topological space and suppose that there is a retraction r of X onto a subspace A. (This means that $r : X \to A$ is a continuous function and that $r(a) = a$ for all $a \in A$.)

(a) Show that r is a quotient map.

(b) Prove that every continuous function from A to a space Z extends to a continuous function from X to Z.

(c) If X is Hausdorff then A is closed.

(d) If $i : A \to X$ is the inclusion map and $a_0 \in A$ then $i_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ is one-to-one.

(e) There is no retraction from a projective plane to a circle.

2. (a) Show that S^{n-1} is a deformation retract of $\mathbf{R}^n - \{0\}$ for $n \geq 1$.

(b) Show that $\mathbf{R}^2$ is not homeomorphic to $\mathbf{R}^n$ where $n \neq 2$.

(c) A theorem known as *Invariance of Domain* asserts that if A and B are subsets of $\mathbf{R}^n$ and $f : A \to B$ is a homeomorphism then $f(Int(A)) \subset Int(B)$. Use Invariance of Domain to deduce that $\mathbf{R}^n$ is homeomorphic to $\mathbf{R}^m$ if and only if $m = n$.

VIII. 1. Let $p : \widetilde{X} \to X$ be a covering space where X is path connected. Show that the sets $p^{-1}(x)$ have the same cardinality for each $x \in X$.

2. Let $p : \widetilde{X} \to X$ be a covering space with $p(\widetilde{x}) = x$. Assume that $\widetilde{X}$ and X are path connected and locally path connected.

(a) State the Lifting Theorem of covering space theory.

(b) If $p_* : \pi_1(\widetilde{X}, \widetilde{x}) \to \pi_1(X, x)$ is surjective what conclusion can be drawn about p?

(c) Give a condition on the space X that ensures that every covering space p, as above, is a homeomorphism.

3. Let $T = S^1 \times S^1$ be a 2-dimensional torus, and let x and y be distinct points in T.

(a) Describe $\pi_1(T - x, y)$.

(b) Describe $\pi_1(T, y)$ by using Van Kampen's Theorem and part (a).