Topology Qualifying Review Exam – August 23, 1991
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Do all problems from #1 – #6.

1. Definitions/Statements.
a) Define: A topological space X is **locally connected**:

b) Let X, Y be topological spaces; let A be a subset of X. Let $f, g : X \to Y$ be two maps which coincide on A. Define what it means to say that the map f **is homotopic to** g **relative to** A:

c) Describe a construction (without proof) of the **Stone-Čech compactification** βX of a completely regular space X, and state its universal property (without proof).

d) Let X and Y be topological spaces; and let Y^X be the set of all maps from X to Y. Define the **compact-open topology** on Y^X.

e) State Urysohn's Lemma:

2. Let $f : X \to Y$ be a continuous map.
Prove or disprove: If X is locally compact, then $f(X)$ is locally compact.

3. Let X and Y be topological spaces; let $\mathcal{F} = \{F_\alpha : \alpha \in J\}$ be a finite family of closed sets of a space X which cover X. Let $f : X \to Y$ be a map whose restriction to each F_α is continuous. Prove or disprove: f is continuous.

4. Let X be a separable (i.e., X contains a countable dense subset) regular space. Prove that every closed set is a G_δ-set.

5. Prove that a closed surjective continuous map is a quotient map.

6. Let X be a metric space; and let A be an arbitrary subset. Recall $d(x, A)$, the distance from x to A, is defined by $d(x, A) = \inf\{d(x, a) | a \in A\}$.
Prove $\overline{A} = \{y \in X | d(y, A) = 0\}$.

Choose three problems from #7 – #10.

7. Let $p : \widetilde{X} \to X$ be a covering space; let $\sigma : I \to X$ be a path. Suppose $f_0, f_1 : I \to \widetilde{X}$ are lifts of the path σ such that $f_0(0) = f_1(0)$. Prove: $f_0 = f_1$. Use "evenly convered neighborhhods" and compactness of I (Uniqueness of Path Lifting).

8. Let p and q be two distinct points in the torus $S^1 \times S^1$. Let X be the disjoint union of $S^1 \times S^1$ and the closed interval $I = [0, 1]$. Identify p with $0 \in I$ and q with $1 \in I$ to get a space Y. Compute the fundamental group of Y. Justify your answer.

9. Let S^2 be the standard 2-sphere and let S^1 be its equator. Prove that S^1 is not a retract of S^2.

10. Let M and N be surfaces with Euler characteristics $\chi(M)$ and $\chi(N)$, respectively. Calculate the Euler characteristic of the connected sum $M\#N$.