ALGEBRA QUALIFYING EXAM

May 1993

0. INSTRUCTIONS

There are four parts to the exam. You are to work two problems from each part.

Partial credit will be given, so work as much of each problem as you can. Each of the eight problems will have equal weight.

I. GROUPS

Work two of the following three problems.

1. If G is a group, let $C(G)$ denote the center of G.
   (a) Prove that if $G/C(G)$ is cyclic, then G must be abelian.
   (b) Give an example of a nonabelian group G for which $G/C(G)$ is abelian.

2. Let p be a prime number, k the field with p elements, $G = \mathrm{G}L_2(k)$. If V is the set of 2×1 vectors over k, then G acts on V by matrix multiplication.
   (a) Verify that $X = V - \{(0,0)^t\}$ is G-invariant.
   (b) Determine the orbit and the stabilizer of $e_1 = (1,0)^t$.
   (c) Use (b) to determine the number of Sylow p-subgroups of G.

3. Recall that if G is a group, then G' denotes the subgroup of G generated by the set $\{xyx^{-1}y^{-1} \mid x, y \in G\}$. If D_n denotes the dihedral group of order $2n$, determine D_n' (*Hint: The answer depends on* n).

II. RINGS

Work two of the following three problems.

1. Let R be a ring, I and J ideals of R. Define $\nu : R \to R/I \times R/J$ by $\nu(a) = (a + I, a + J)$.
   (a) Verify that ν is a ring homomorphism.
   (b) Prove that ν is injective iff $I \cap J = (0)$.
   (c) Prove that ν is surjective iff $I + J = R$.

2. Let R be a ring, $R[x]$ the ring of polynomials over R, I an ideal of $R[x]$. Prove that if I contains a monic polynomial, then $R[x]/I$ is a finitely generated R-module.

3. Let A be a $\mathbb{C}$-subalgebra of the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. Prove that if M is a maximal ideal of $\mathbb{C}[x_1, \ldots, x_n]$, then $M \cap A$ is a maximal ideal of A. (*Hint: A need* **not** *be a finitely generated* $\mathbb{C}$*-algebra*).

III. FIELDS

Work two of the following three problems.

1. Let $\overline{k}$ denote the algebraic closure of a field k. If E/F is an extension of fields, prove that $\overline{E} \approx \overline{F}$ iff E is algebraic over F.

2. Let E/F be a finite Galois extension, $G = \mathrm{A}ut(E/F)$ the Galois group of the extension. For each $\alpha \in E$, let $S_\alpha = \{g \in G \mid g(\alpha) = \alpha\}$. Prove that α is a primitive element for E iff $S_\alpha = \langle 1 \rangle$.

3. Compute the Galois group over $\mathbb{Q}$ of **one** of the following quartic polynomials:
   (a) $f(x) = x^4 - 8x^2 - 2$;
   (b) $f(x) = x^4 - 4x - 2$. (*Hint: In both cases, the number of real roots of* $f(x)$ *is useful information*).

IV. MODULES

Work two of the following three problems.

1. Let R be a ring, I a right ideal of R, $\pi : R \to R/I$ the canonical projection onto the quotient module. Suppose that there exists an R-module map $\lambda : R/I \to R$ which satisfies $\pi\lambda(a + I) = a + I$ for all $a \in R$. Prove that:
   (a) $I = eR$, where $e^2 = e$;
   (b) both I and R/I are projective R-modules.

2. Let k be a field, A an $n \times n$ matrix over k, V the $k[t]$-module defined by A. Prove that if the characteristic polynomial of A is irreducible in $k[t]$, then V is a simple $k[t]$-module.

3. Let K be a commutative ring, I an ideal of K, M a left K-module. Prove that the K/I-modules $K/I \bigotimes_K M$ and M/IM are isomorphic.