Department of Mathematics
M. A. Comprehensive/Ph.D. Qualifying Exam in Analysis
May, 1994

Directions: Work as many problems as you can; you have three hours. You can appeal to results proved in your courses (unless you are asked to prove one of those results), but such results should be clearly cited. $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the positive integers, rational numbers, and real numbers, respectively.

1. (a) Let $S \subseteq \mathbb{R}$ be closed, let $f: S \to \mathbb{R}$ be continuous on S, and let $(x_n) \subseteq S$ be a Cauchy sequence. Prove that $(f(x_n))$ is a Cauchy sequence.
(b) Show by example that the result of (a) can fail for continuous functions $f: S \to \mathbb{R}$ if S is not closed in $\mathbb{R}$.

2. Construct a closed set $F \subseteq \mathbb{R}$ that has infinite Lebesgue measure, but contains no nonempty open intervals.

3. (a) Let M be a σ-algebra of subsets of a set X, let Y be a set, and let $f: Y \to X$ be a function. Prove that the family of sets
$$\mathrm{N} = \{f^{-1}(E) \mid E \in \mathrm{M}\}$$
is a $\sigma-$algebra on Y.
(b) Let $f: \mathbb{R} \to \mathbb{R}$ be continuous; prove that for every Borel set $Q \subset \mathbb{R}$ the inverse image $f^{-1}(Q)$ is also a Borel set.

4. State and prove Egoroff's theorem.

5. (a) Give a precise statement of the Monotone Convergence Theorem for the Lebesgue integral on an abstract measure space (X, M, μ).
(b) Consider the measure space $(\mathbb{N}, \mathrm{P}(\mathbb{N}), \lambda)$, where $\mathrm{P}(\mathbb{N})$ is the power set (i. e., set of all subsets) of $\mathbb{N}$ and λ is the counting measure. If $f: \mathbb{N} \to [0, \infty)$ is an arbitrary function, then give a precise proof that f is measurable and
$$\int_{\mathbb{N}} f \, d\lambda = \sum_{n=0}^{\infty} f(n),$$
where the sum of the series is set to ∞ if it diverges.

6. (a) Let (X, M, μ) be a measure space and let (f_n) be a sequence of nonnegative measurable functions of X into $[0, \infty)$ that converges in measure to a function $f: X \to [0, \infty)$ and satisfies $\lim_{n\to\infty} \int_X f_n \, d\mu = A$; prove that $\int_X f \, d\mu \leq A$. (Hint: use Fatou's lemma.)
(b) Give an example where strict inequality holds in the conclusion of part (a).

7. If $f, g: [a, b] \to \mathbb{R}$ are absolutely continuous functions on $[a, b]$, then prove that the product fg defined by $(fg)(x) = f(x)g(x)$ is absolutely continuous on $[a, b]$.

8. (a) State Hölder's inequality for functions defined on an abstract measure space (X, M, μ).
(b) Let (X, M, μ) be a measure space and let $r, s \in \mathbb{R}$ be such that $r > 1, s > 1$. Find $\alpha > 0$ so that if $f \in L^r(\mu)$ and $g \in L^s(\mu)$, then $|fg|^\alpha \in L^1(\mu)$.

9. Let $g: \mathbb{R} \to \mathbb{R}$ be defined by $g(t) = t^3$, let $h: \mathbb{R} \to \mathbb{R}$ be defined by $h(t) = t^5$, and let λ_g and λ_h denote the Lebesgue-Stieltjes measures associated to g and h, respectively (view these measures as defined on the family B of Borel subsets of $\mathbb{R}$). Show that both of the relations $\lambda_g \ll \lambda_h$ and $\lambda_h \ll \lambda_g$ are true and compute the corresponding Radon-Nikodym derivatives $d\lambda_h/d\lambda_g$ and $d\lambda_g/d\lambda_h$.

10. Let $(\mathbb{N}, \mathrm{P}(\mathbb{N}), \lambda)$ be as in problem 5(b) and let $([0, 1], \mathrm{M}, \mu)$ be the measure space where M is the family of Lebesgue measurable subsets of $[0, 1]$ and μ is the Lebesgue measure. Define $f: \mathbb{N} \times [0, 1] \to [0, \infty)$ by $f(n, x) = (ne^{-nx})/2^n$. Prove that f is measurable with respect to the product σ-algebra $\mathrm{P}(\mathbb{N}) \times \mathrm{M}$ and evaluate
$$\int_{\mathbb{N}\times[0,1]} f \, d(\lambda \times \mu);$$
justify your steps and point out any standard theorems that you use.