TOPOLOGY QUALIFYING EXAM

May 1994

Work as many problems as you can. Give complete explanations, but try not to waste time verifying obvious details.

1. a. Show that every second countable space contains a countable dense subset.
   b. Conversely, show by example that a space with a countable dense subset need not be second countable.

2. Give $\prod_{i=1}^{\infty} \mathbb{R}$ the product topology, and let $f\colon X \longrightarrow \prod_{i=1}^{\infty} \mathbb{R}$ be a function. Let π_i denote projection to the i^{th} coordinate.
   a. Suppose f is a closed map, and that $\pi_i \circ f(X)$ is bounded for all i. Show that $f(X)$ is compact.
   b. Show by example that a function satisfying the hypotheses in part a. need not be continuous.

3. a. Define the Lebesgue number of an open covering and state the Lebesgue number lemma.
   b. Let D denote the closed unit disk in $\mathbb{R}^2$ and let $X = D \setminus \{0\}$. Show that the conclusion of the Lebesgue number lemma does not hold for X.

4. a. Prove that if $f : X \to Y$ is continuous, and X is connected, then $f(X)$ is a connected subspace of Y.
   b. Give an example of a continuous map $f\colon X \longrightarrow Y$ between compact connected spaces X and Y such that for all $y \in Y$, the set $f^{-1}(\{y\})$ is not connected.

5. Let $GL(2, \mathbb{R})$ denote the set of non-singular 2×2 matrices with real entries. Let $SL(2, \mathbb{R}) \subset GL(2, \mathbb{R})$ denote the matrices with determinant equal to 1. Give $GL(2, \mathbb{R})$ and $SL(2, \mathbb{R})$ topologies by regarding them as subsets of $\mathbb{R}^4$ with the subspace topology.
   a. Determine whether or not $GL(2, \mathbb{R})$ is compact.
   b. Determine whether or not $SL(2, \mathbb{R})$ is compact.

6. a. Suppose X is regular and second countable. Show that X is normal.
   b. Suppose X has the property that given any closed set $A \subset X$, there exists a continuous function $f\colon X \longrightarrow \mathbb{R}$ such that $f(a) = 0$ for all $a \in A$, and $f(x) > 0$ for all $x \in X \setminus A$. Show that X is normal.

7. Suppose X is a compact Hausdorff space, and that A is a nonempty closed subset. Show that X/A, the space obtained by identifying A to a point, is homeomorphic to $\widehat{X \setminus A}$, the one-point compactification of $X \setminus A$.

Do **two** of the following three problems.

8. Let X denote the space obtained by gluing two tori along a common meridian as indicated in the figure on the left. (X can also be described as the space you obtain by rotating the figure on the right, which is the union of two circles that meet in a single point, around the z axis.)

   Use Van Kampen's theorem to compute $\pi_1(X)$.

9. Let $C(X, \mathbb{R}) = \{f : X \to \mathbb{R} \mid f \text{ is continuous}\}$. Prove that $C(X, \mathbb{R})$ is contractible. (*Hint:* Show that $F : C(X, \mathbb{R}) \times I \to C(X, \mathbb{R})$ given by $(f, t) \mapsto t \cdot f$ is continuous.)

10. Let $\phi : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ be given by $\phi(x, y) = (2x, 2y)$ and let $G = \langle \phi \rangle$, the group generated by ϕ.
    a. Describe $X = \mathbb{R}^2 \setminus \{0\}/G$. (It is a familiar space.)
    b. Show that $\pi : \mathbb{R}^2 \setminus \{0\} \to X$ is a covering map, and determine the group of covering transformations.