ALGEBRA QUALIFYING EXAM

May 1995

INSTRUCTIONS

There are four parts to the exam. Work the indicated number of problems from each part.

Partial credit will be given, so work as much of each problem as you can. Each problem will have equal weight.

I. GROUPS

Work three of the following five problems.

1. Let G be a group and N a normal subgroup of G with $(G : N) < \infty$. Let p be prime factor of $(G : N)$. Prove that there is a subgroup H, $N \leq H \leq G$ with $(G : H)$ relatively prime to p and $(N : H)$ a power of p.

2. Let G be a group of order p^r, p a prime. Prove that every proper maximal subgroup of G is normal.

3. Let G be a finite group and X a finite set on which G acts transitively. Prove that

$$\bigcup_{x \in X} G_x \neq G.$$

4. Let

$$G = \varprojlim_{i \in I} (G_i)$$

be an inverse limit such that each group G_i is abelian. Prove that G is abelian.

5. Let (F, j) be a free group on the set X. Let X_0 be a subset of X, let (F_0, j_0) be a free group on X_0, let $h_0 : X_0 \to X$ be the inclusion map and let $h : F_0 \to F$ be the induced group homomorphism. Prove that there is a group homomorphism $k : F \to F_0$ such that $kh = id_F$.

II. RINGS

Work two of the following three problems

1. Let P be a prime ideal of $\mathbb{Z}[X]$ and suppose $P \cap \mathbb{Z} = 0$. Prove that P is principal.

2. Let p be a prime integer. Prove that $X^{p-1} + X^{p-2} + \ldots X + 1$ is irreducible in $\mathbb{Z}[X]$.

3. Let R be a UFD. Suppose that for every pair of relatively prime non–units $p, q \in R$ the ideal $Rp + Rq$ is principal. Prove that for every pair of elements $a, b \in R$ the ideal $Ra + Rb$ is principal.

III. FIELDS

Work problem 1 and any two of the others.

1. Let $k = \mathbb{Q}(\pi) \subset \mathbb{R}$. Set $f = x^3 - \pi \in k[x]$. Let $E \subset \mathbb{C}$ be a splitting field for f over k. Use Galois theory to find all the fields F with $k \subseteq F \subseteq E$.

2. Suppose that E, k and F are subfields of the field K with $k \subseteq E \cap F$. Assume that $E \supseteq k$ is algebraic. Prove that $EF \supseteq F$ is algebraic.

3. Let k be a field and let $E = k(X)$ be a pure transcendental extension of k. Let $f \in E - k$. Prove that $[k(f) : k] = \infty$.

4. Let $E \supset k$ be a Galois extension of fields with $[E : k] = 2$ and characteristic of k $\neq 2$. Prove that there exists $\alpha \in E - k$ with $\alpha^2 \in k$.

IV. VECTOR SPACES

Work one of the following three problems.

1. Let V be a finite dimensional vector space over the field F. Let $W_1, \ldots W_n$ be subspaces of V such that $V = \cup_i W_i$. Prove that $W_i = V$ for some i.

2. Let V be a vector space over the field E of dimension n. Let $k \supset E$ be a subfield such that $[E : k] = m$. Prove that V is a vector space over k of dimension mn.

3. Let A be an n by n matrix over $\mathbb{Q}$. Show that the entries of the Jordan canonical form of A all lie in a field $E \subset \mathbb{C}$ such that $[E : \mathbb{Q}] \leq n!$