REAL ANALYSIS I SPRING 1995

M.A. Comprehensive/Ph.D. Qualifying Exam

In order to get full credit, you must provide complete justifications of your answers. Note that this is the first part of a two part exam. You have three hours to complete as many questions as you can. Please separate your answers to the different parts of the exam.

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1. For $0 < \alpha \leqslant 1$, describe the construction of the Cantor-like set $S^{(\alpha)}$. Prove that $S^{(\alpha)}$ is perfect and nowhere dense. Calculate the total length of the open intervals that comprise $[0,1] \setminus S^{(\alpha)}$. Prove that $S = \bigcup_{n=1}^{\infty} S^{(1/n)}$ is not closed. Explain why S is measurable and calculate $m(S)$.

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2. Let S be the set of dyadic rational numbers in $[0,1]$, that is,

$$S = \{\frac{p}{2^n} : p = 0, \dots, 2^n \quad \text{and} \quad n = 0, 1, \dots\}$$

and let $\epsilon > 0$. Explain how to cover S with countably many intervals whose total length is less than ϵ. Now assume that the finite set, $\{I_n\}_{n=1}^{N}$, of intervals covers S, that is $S \subset \bigcup_{n=1}^{N} I_n$. Prove that $\sum_{n=1}^{N} \ell(I_n) \geqslant 1$.

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3. Let $E_1, E_2, \dots$ be disjoint measurable sets. Prove that

$$m^*\Big(\bigcup_{n=1}^{\infty} E_n\Big) \geqslant \sum_{n=1}^{\infty} m^*(E_n).$$

Explain how to construct disjoint sets $S_1, S_2, \dots$ with

$$m^*\Big(\bigcup_{n=1}^{\infty} S_n\Big) < \sum_{n=1}^{\infty} m^*(S_n).$$

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4. State Lebesgue's Dominated Convergence Theorem. Use it (with clear justifications) to calculate the following limits:

$$\text{(a)} \lim_{n\to\infty} \int_0^1 \frac{1+nx}{(1+x)^n}\, dx \qquad \text{(b)} \lim_{n\to\infty} \int_0^{\infty} \frac{dx}{(1+x/n)^n x^{1/n}}$$

5. State Fatou's Lemma, and show that the sequence $\{f_n\}_{n=1}^{\infty}$, defined by

$$f_n(x) = \begin{cases} \dfrac{1}{n}, & \text{for } x \in [1, n] \\ 0, & \text{for } x \in \mathbb{R}\backslash[1, n] \end{cases}$$

for each $n = 1, 2, \ldots$, provides an example where strict inequality holds.

(a) Use Fatou's Lemma to prove that

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n,$$

where f_n is measurable and non-negative for each n.

(b) Use the result of part (a) to show that if $p > -1$ then

$$\int_0^1 \frac{x^p \ln x}{1-x}\, dx = -\sum_{n=1}^{\infty} \frac{1}{(p+n)^2}.$$

6.

(a) Put

$$f(x) = \begin{cases} x^p \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Show that f has bounded variation on $[0, 1]$ if $p \geqslant 2$ and that f is not of bounded variation on $[0, 1]$ if $0 < p \leqslant 1$.

(b) Let E be measurable with $0 < m(E) < \infty$. Prove that

$$\lim_{h \to 0^+} \frac{m(E \cap (x-h, x+h))}{2h} = \chi_E(x)$$

for almost almost all x.