Qualifying/Review Exam in Topology
5–17–96

Work problems from each of the four groups I, II, III, IV as indicated. Throughout X and Y will denote arbitrary topological spaces and, unless otherwise specified, the real line $\mathbb{R}$ is assumed to be endowed with the Euclidean topology. Also, $\mathbb{Q}$ denotes the set of rational numbers, $\mathbb{J} = \mathbb{R} - \mathbb{Q}$ and $\mathbb{C}$ is the set of complex numbers.

I. For the problems in this group, carefully define each topological term which appears, then verify the statement.

1. The collection $\mathcal{B} = \{ (a,b) \mid a,b \in \mathbb{R} \} \cup \{ \{x\} \mid x \in \mathbb{J} \}$ forms a basis for a topology on the real line $\mathbb{R}$.
2. A compact Hausdorff space is regular.
3. A contractible space X has trivial fundamental group $\pi_1(X, x_0)$ for any $x_0 \in X$.

II. Work 4 of the 6 problems from this section.

4. Let C be a component of X.
   a. Show that C is closed.
   b. If X is locally connected then C is open.
   c. In general C may not be open.
5. A function $f : X \to Y$ is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for each $A \subset X$.
6. Show that every nonconstant path in the Euclidean plane $\mathbb{R}^2$ contains a point one of whose coordinates is rational and a point one of whose coordinates is irrational.
7. Show that $\pi_1(X \times Y, x_0 \times y_0)$ is isomorphic to $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.
8. A connected metric space with more than one element is uncountable.
9. Let X be the subspace of the Euclidean plane which is the union of $\{0\} \times [-1,1]$ and $\{(x,y) \in \mathbb{R}^2 \mid 1 \le x^2 + y^2 < 2\}$.
   a. Determine the fundamental group of X.
   b. Briefly explain why the "figure eight" is a deformation retract of X.

III. Determine whether the following are true or false. In each case, provide either a proof or a counterexample as appropriate. (Be sure to explain the counterexample.)

10. The topology on $\mathbb{R}$ given in problem # 1 is metrizable.
11. Let X be a first countable space and let A be a closed subset of X. Then $x \in A$ if and only if there is a sequence x_n in A which converges to x.
12. Let $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$. The inclusion of S^1 into $\mathbb{C} - \{(\frac{1}{2}, 0)\}$ is a homotopy equivalence.
13. A locally compact Hausdorff space is regular.

IV. Work 2 of the 3 problems in this section.

14. A function $f : X \to \mathbb{R}$ is said to be *lower semicontinuous* if $f^{-1}((b, \infty))$ is open for each $b \in \mathbb{R}$. Let $\{ f_\lambda \mid \lambda \in \Lambda \}$ be a family of continuous functions from X to $\mathbb{R}$ and suppose that $\{ f_\lambda(x) \mid \lambda \in \Lambda \}$ is bounded above for each $x \in X$. Define a function $M : X \to \mathbb{R}$ by $M(x) = \sup\{ f_\lambda(x) \mid \lambda \in \Lambda \}$.
    a. Show that M is lower semicontinuous.
    b. Show that M may fail to be continuous.
15. Let X be a compact Hausdorff space. Suppose that X has an open cover $\mathcal{U}$ where each $U \in \mathcal{U}$ is metrizable.
    a. Show that X is metrizable.
    b. Show by example that if either the assumption that X is compact or the assumption that each $U \in \mathcal{U}$ is open is removed then X may fail to be metrizable.
16. Let $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$ be covering maps.
    a. Show that $p \times q$ is a covering map from $\widetilde{X} \times \widetilde{Y}$ to $X \times Y$.
    b. Show that the group of covering transformations $A(\widetilde{X} \times \widetilde{Y}, p \times q)$ of $p \times q$ is isomorphic to $A(\widetilde{X}, p) \times A(\widetilde{Y}, q)$.
    c. The covering map $p \times q$ is regular if and only if both p and q are regular.