ALGEBRA

Qualifying Exam

May 1999

Instructions: Do as many problems as you can. Please justify your answers and show your work. All rings are associative with 1.

1. Prove or disprove:

(a) **If every subgroup of a group G is normal, then G is abelian.**
(b) **If D is PID then so is $D[x]$.**
(c) **If $\mathbb{E}/\mathbb{K}$ and $\mathbb{K}/\mathbb{F}$ are finite Galois extensions then $\mathbb{E}/\mathbb{F}$ is also Galois.**

2. Let p be a prime number and G be a group with $|G| = p^n$. Show:

(a) **The center of G is non-trivial.**
(b) **For each $k \leq n\, G$ has a normal subgroup H of order p^k.**

(3) Prove:
(a) **If G is a group of order 12 then G is not simple.**
(b) **If G is a group of order p^2q, where p and q are distinct primes, then G is not simple.**

4. If I and J are ideals of a ring R and $I+J=R$, prove that $R/I\cap J$ is isomorphic to $R/I \bigoplus R/J$.

5. Show that an element a of a commutative ring A is nilpotent (i.e., $a^k = 0$ for some positive integer k) if and only if a is contained in every prime ideal of A.

6. If M is a finitely generated module over a Noetherian ring and $f : M \to M$ is an epimorphism, prove that f is an automorphism.

7. Classify, up to similarity, all 3×3 matrices T satisfying $T^3 = T$ over an arbitrary field $\mathbb{F}$.

8. Prove:

(a) If G is a finite non-cyclic abelian group then there is a positive integer $k < |G|$ such that $g^k = e$ (e is the identity of G), for every $g \in G$.

(b) Every finite multiplicative subgroup of the group of non- zero elements of a field is cyclic.

9. Let $\mathbb{E}$ be the splitting field of $x^4 - x^2 + 2$ over $\mathbb{Q}$. Find all intermediate fields $\mathbb{K}$ between $\mathbb{E}$ and $\mathbb{Q}$.

10. Let $\mathbb{E}$ be a finite extension of $\mathbb{F}_q$ (the finite field with q elements). Show that $\mathbb{E}$ is Galois over $\mathbb{F}_q$ and $Gal(\mathbb{E}/\mathbb{F}_q)$ is cyclic.