Algebra Qualifying Exam, September 1999

Instructions: Do as many problems as you can. Please justify your answers and show your work. All rings are associative with 1.

1. Prove or disprove:
(a) If H and K are subgroups of finite index in a group G then $H \cap K$ also has finite index in G.
Let $f : R \to S$ be a ring homomorphism.
(b) If I is a prime ideal of S then $f^{-1}(I)$ is a prime ideal of R.
(c) If I is a maximal ideal of S then $f^{-1}(I)$ is a maximal ideal of R.

2. Let K be a normal subgroup of a finite subgroup G. Assume $n = [G : K]$ is relatively prime to $|K|$. Show that $K = \{x^n : x \in G\}$. What if K is not normal in G?

3. Prove: (a) If G is a group of order 12 then G is not simple.
(b) If G is a group of order p^2q, where p and q are distinct primes, then G is not simple.

4. If I and J are ideals of a ring R and $I + J = R$, prove that R/IJ is isomorphic to $R/I \times R/J$.

5. Show that an element a of a commutative ring A is nilpotent (i.e., $a^k = 0$ for some positive integer k) if and only if a is contained in every prime ideal of A.

6. If M is a finitely generated module over a Noetherian ring and $f : M \to M$ is an epimorphism, prove that f is an automorphism.

7. prove that any square matrix (over any field) is similar to its transpose.

8. Prove: (a) If G is a finite non-cyclic abelian group then there is a positive integer $k < |G|$ such that $g^k = e$ (e is the identity of G), for every $g \in G$.
(b) Every finite multiplicative subgroup of the the group of non-zero elements of a field is cyclic.

9. Let $\mathbb{E}$ be the splitting field of $x^4 - 4x^2 + 2$ over $\mathbb{Q}$. Find all intermediate fields $\mathbb{K}$ between $\mathbb{E}$ and $\mathbb{Q}$.

10. Let $\omega_n = e^{2\pi i/n}$.
(a) Describe the action of $Gal(\mathbb{Q}(\omega_n)/\mathbb{Q})$ on $\mathbb{Q}(\omega_n)$.
(b) Show that any subfield of $\mathbb{Q}(\omega_n)$ is Galois over $\mathbb{Q}$.