REAL ANALYSIS

Qualifying Examination

Fall 1999

1. Let (X, B, μ) be a measure space, and $< E_i >$ be a sequence of sets in B. Prove that

$$\mu(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu E_i$$

2a. State Hahn-Banach Theorem.

2b. Let X be a normed vector space and X^{**} be the dual of the dual of X. Prove by an application of Hanh-Banach theorem that there is a natural isomorphism from X into a linear subspace of X^{**}.

3a. Prove that if $< f_n >$ is an equicontinuous sequence of mapping from a metric space X to a complete metric space Y, and if the sequences $< f_n(x) >$ converge for each x of a dense subset D of X, then $< f_n >$ converges at each point of X, and the limit function is continuous.

3b. State Ascoli-Arzelá Theorem.
Show that the Ascoli-Arzelá Theorem may fail if the various hypotheses are dropped for the following sequences of functions:

3c. $f_n(x) = x + n$ for $x \in [0, 1]$.

3d. $f_n(x) = x^n$ for $x \in [0, 1]$.

4. State and Prove Riesz representation theorem in the classical Banach spaces: Let F be a bounded linear functional on $L^p, 1 \leq p < \infty$. Then there exists a **unique** function g in L^q such that

$$F(f) = \int fg.$$

We also have $||F|| = ||g||_q$.

5. Prove the Projection Theorem in a Hilbert space:
Let S be a closed subspace of a Hilbert space H, and $S^\perp$ be the set of elements orthogonal to every element of S. Then for every $x \in H$ we have $x = y + z$ where $y \in S$ and $z \in S^\perp$.

6a. State Fubini's Theorem

6b. Let

$$f(x,y) = \begin{cases} y^{-2} & \text{if } 0 < x < y < 1, \\ -x^{-2} & \text{if } 0 < y < x < 1, \\ 0 & \text{otherwise if } 0 \le x \le 1, 0 \le y \le 1. \end{cases}$$

Compute

$$\int_0^1 \int_0^1 f(x,y)dydx \text{ and } \int_0^1 \int_0^1 f(x,y)dydx$$

.

(Hint $\int_0^1 f(x,y)dx = \int_0^y \frac{dx}{y^2} - \int_y^1 \frac{dx}{x^2}$)

6c. Is it true that $\int_0^1 \int_0^1 f(x,y)dydx = \int_0^1 \int_0^1 f(x,y)dxdy$? Is Fubini's Theorem applicable?
Let $< X, p >$ be a metric space.
Prove that the following four statements are equivalent:

7a. $< X, p >$ is compact.

7b. $< X, p >$ is sequentially compact.

7c. $< X, p >$ has the Bolzano-Weierstrass property.

7d. $< X, p >$ is complete and totally bounded.

8a. Prove Young's inequality

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

where $a, b \ge 0, 1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1$.

8b. Use 8a, Young's inequality to give a proof of the Höder inequality.

9a. If $f(x)$ is a function of bounded variation, then is it necessary that $f(x)$ has to be bounded?

9b. Let $f(x) = xcos(\pi/2x)$ if $x \neq 0, f(0) = 0$. Prove or disprove that f is of bounded variation in [0,1].

10 Prove the following generalization of Lebesgue Convergence Theorem: Let $< g_n >$ be a sequences of integrable functions which converges a.e. to an integrable function g. Let $< f_n >$ be a sequences of measurable functions such that $|f_n| \leq g_n$, and $< f_n >$ converges to f a.e. If

$$\int g = lim \int g_n.$$

then

$$\int f = lim \int f_n.$$