ALGEBRA

Qualifying Exam

May 2000

Instructions: Do as many problems as you can and show all your work.

1. Let G be a group of order 12. Show that either G has a normal Sylow 3-subgroup or $G \cong A_4$.

2. Prove that if H and K are finite groups of G, whose orders are relatively prime, then $|HK| = |H| \cdot |K|$.
(Here $HK = \{hk | h\epsilon H, k\epsilon K\}$.)

3.) Let G be a group with $|G| = p^n$. Show for each $k \leq n$, G has a normal subgroup of order p^k.

4.) Give an example of a ring with no maximal ideals.

5.) Show that if R is a commutative ring with identity and $R[x]$ is a PID, then R is a field.

6.) Let A, B be $n \times n$ matrices over $\mathbb{Q}$. Suppose there is an invertible $n \times n$ complex matrix C with $C\,A\,C^{-1} = B$. Prove that there is an invertible $n \times n$ rational matrix D with $D\,A\,D^{-1} = B$.

7.) Classify, up to similarity, all 3×3 matries T satisfying $T^3 = T$ over $\mathbb{C}$.

8.) Let K be the splitting field of $x^3 - 2$ over $\mathbb{Q}$. Determine all the intermediate fields E between K and $\mathbb{Q}$.

9.) Prove that it is impossible to construct the regular 9-gon by straightedge and compass.