TOPOLOGY

Qualifying Examination

May 2000

For each question, either supply a proof or counterexample. Giving a complete solution to one problem is better than two half solutions to two problems. Use a separate sheet for each problem.

(1) If (X, d) is a complete metric space, recall that a map $f \cdot X \rightarrow X$ is called a contraction if there exists $\alpha < 1$ s .t . $d(f(x), f(y)) \leq \alpha d(x, y)$, for all $x, y \,\epsilon\, X$. Prove that if f is a contraction, then f has a unique fixed point in X. What if $\alpha = 1$ above? Is there still a fixed point?

(2) Let A and B be connected subsets of the topological space X. Prove that if $A \cap \overline{B} \neq \phi$, then $A \cup B$ is connected.

(3) Let $I = [0, 1]$ and consider $C(I, I)$, the space of continuous functions from I to I. Is $C(I, I)$ an equicontinuous family?

(4) Let X be the reals with the Lower limit topology. Prove that X has a countable dense subset, but does not have a countable basis.

(5) Let $Y = \prod_{i=1}^{\infty} \mathbb{R}$ with the product topology. Consider the subset $X = \{(x_n) \epsilon Y : \sum x_n$ converge and $\sum x_n = 1\}$. Prove or disprove that X is compact.

(6) Prove that a connected manifold is path connected. (Recall: A manifold is a Hausdorff topological space which has a countable basis and is locally homeomorphism to an open subset of $\mathbb{R}^m$).

(7) Is the open ball centered at the origin in $\mathbb{R}^n$ homeomorphic to $\mathbb{R}^n$? (If yes, exhibit a homeomorphism).

(8) Classify all the coverings of S^1. Make sure you explicity write down the coverings.

(9) Is there a retraction of S^2 onto its equator?

(10) Let G be a group of homeomorphisms of S^n. Assume G is infinite. Can G act properly discontinuously on S^n? If yes, give an example; if not, prove it.