Real Analysis Qualifying Exam, May 2001

Instructions: This is an open-book examination.

[10] **1.** Let $f(x)$ be a continuous function from $[0,1]$ to $[0,1]$. Prove that there is a $x_0 \in [0,1]$ such that $f(x_0) = x_0$.

[20] **2.** Let E be a Lebesgue measurable subset of R with $0 < m(E) < \infty$. Let χ_E denote the characteristic function of E.

[10] a) Prove that the function

$$\phi(x) = \int_R \chi_E(y) \cdot \chi_E(x+y)dy$$

is continuous at every point x in R.

[10] b) Use the result in part a) to show the following: Let $F = \{x - y \ : \ x, y \in E\}$, there exists a $\delta > 0$ such that $(-\delta, \delta) \subset F$.

[20] **3.**

[10] a) If $f(x)$ is of bounded variation on $[0,1]$, show that for any $a \in (0,1)$ the limit of $f(x)$ exists as $x \to a-$.

[10] b) If $f(x)$ is absolutely continuous on $[0,1]$, show that

$$T_0^1(f) = \int_0^1 |f'| \ .$$

[20] **4.** Prove that $L^\infty[0,1]$ is complete under $||\cdot||_\infty$. Does such a norm induce an inner product in $L^\infty[0,1]$ (or: is $L^\infty[0,1]$ a Hilbert space?)

[10] **5.** Let $p \in [1,\infty)$ and E be a measurable set in R. Prove that $\lim_{n\to\infty} \int_E |f_n - f|^p = 0$ if and only if f_n converges almost everywhere to a function $f \in L^p(E)$ and $\lim_{n\to\infty} \int_E |f_n|^p - \int_E |f|^p = 0$.

[20] **6.** We define l^2 space as the collection $l^2 = \{\overline{x} = \{x_i\}_{i=1}^{\infty} \mid \sum_{i=1}^{\infty} |x_i|^2 < +\infty\}$. We define the norm for an element $\overline{a} = \{a_i\}$ in l^2 by

$$\|\overline{a}\|_2^2 = \sum_{i=1}^{\infty} |a_i|^2.$$

It can be proved that l^2 is a Hilbert space.

A sequence $\{\overline{a}^n\}$ in l^2 space is said to converge weakly to an element $\overline{a}$ in l^2 if for every $\overline{b} \in l^2$

$$\langle \overline{a}^n, \overline{b} \rangle \to \langle \overline{a}, \overline{b} \rangle \qquad \text{as} \qquad n \to \infty,$$

where $\langle \cdot, \cdot \rangle$ is the inner product induced from the l^2 norm.

[10] a) Prove that if $\overline{a}^n$ converges to $\overline{a}$ in l^2 (usually we call such convergence "strongly convergent" comparing with "weakly convergent"), then $\overline{a}^n$ converges weakly to $\overline{a}$.

[10] b) Find a sequence that weakly converges to $\overline{0}$ in l^2, but does not strongly converge to $\overline{0}$.

[100] **Total Marks**