Topology Qualifying Examination
August 13, 2001

Instructions: Try to give complete arguments, but do not spend excessive time verifying obvious details, especially when giving examples. Apply major theorems when possible. In the "A" section, try to do all of the problems. In the "B" section, do as many problems as you can; it is not expected that you will be able to complete all of them.

A. Try to do all of the problems in this section.

I. Let X be a topological space and A a subset of X. For this problem, assume that the *closure* $\overline{A}$ of A is defined to be a closed subset of X with the properties that $A \subseteq \overline{A}$, and if C is any closed subset of X with $A \subseteq C$, then $\overline{A} \subseteq C$.

1. Prove that $\overline{A}$ equals the intersection of all closed subsets of X that contain A.
2. Prove that $x \in \overline{A}$ if and only if every open neighborhood of x contains a point of A.

II. Let X be the Hawaiian earring $\cup_{n=1}^{\infty} C_n$, where C_n is the circle in the xy-plane with center $(\frac{1}{n}, 0)$ and radius $\frac{1}{n}$. Its topology is the subspace topology as a subset of the xy-plane. Define a function $g\colon X \to X$ by $g(x, y) = (nx, ny)$ for $(x, y) \in C_n$. Prove that g is not continuous.

III. Let X_α, $\alpha \in \mathcal{A}$ be a collection of topological spaces, and let $\prod X_\alpha$ be their product, with the product topology. Recall that a basic open set in the product topology can be written as $U_{\alpha_1} \times \cdots \times U_{\alpha_n} \times \prod_{a \notin \{\alpha_1, \ldots, \alpha_n\}} X_\alpha$, where each U_{α_i} is an open subset of X_{α_i}.

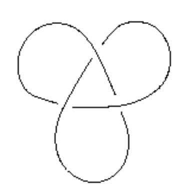
1. Let $\pi_\beta\colon \prod X_\alpha \to X_\beta$ denote the projection function to one of the factors. Prove that π_β is continuous.
2. Prove that a function $f\colon Z \to \prod X_\alpha$ is continuous if and only if each of its coordinate functions $\pi_\beta \circ f$ is continuous.
3. Suppose that each X_α is homeomorphic to the line $\mathbb{R}$ (with the standard topology), and that the index set $\mathcal{A}$ is uncountable. Prove that $\prod X_\alpha$ is not first countable, by proving that there is no countable local basis at the point (x_α) which has all coordinates $x_\alpha = 0$.

IV. Let $X = \prod_{i=1}^{\infty} \mathbb{R}$, the product of countably many copies of the real line (where $\mathbb{R}$ has the standard topology and the product has the product topology). Let $A = \{\, a_n \mid n \in \mathbb{N} \,\}$ be a set of real numbers, and for each $n \in \mathbb{N}$, let $x_n \in X$ be the point $(a_n, \ldots, a_n, 0, 0, \ldots)$, where the first n coordinates are a_n and all other coordinates are 0. Suppose that $T\colon X \to \mathbb{R}$ is a continuous function. Prove that if A is a bounded subset of $\mathbb{R}$, then $\{\, T(x_n) \mid n \in \mathbb{N} \,\}$ is a bounded subset of $\mathbb{R}$.

V. Define what it means to say that X is a *metric space,* and define the *metric toplogy* induced by a metric. Give examples of:

1. A metric space each of whose ϵ-balls has uncountably many path components.
2. A metric space each of whose ϵ-balls has a countably infinite number of path components, each of which is homeomorphic to the open interval $(0, 1)$.

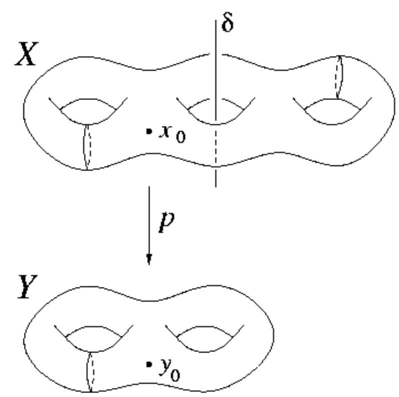
VI. The figure to the right shows a trefoil knot K in 3-dimensional space $\mathbb{R}^3$. It is the image of an imbedding $i\colon \mathrm{S}^1 \to \mathbb{R}^3$, where S^1 is the unit circle in the complex plane $\mathbb{C}$. Define $f\colon K \to K$ by $f(i(z)) = i(z^2)$. Prove that there exists a continuous function $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ whose restriction to K is f. (Hint: It is sufficient to find an extension for each of the three coordinate functions.)



B. In this section, do as many problems as you can; it is not expected that you will be able to complete all of them.

VII. Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of natural numbers, with the discrete topology, and let $\mathbb{R}$ be the real numbers with the standard topology. Let $\mathrm{C}(\mathbb{N}, \mathbb{R})$ be the space of continuous functions from $\mathbb{N}$ to $\mathbb{R}$ (that is, all functions from $\mathbb{N}$ to $\mathbb{R}$), with the compact-open topology. Let $\mathcal{B}$ be the subspace of $\mathrm{C}(\mathbb{N}, \mathbb{R})$ consisting of the bounded functions, and let $\mathcal{U}$ be the subspace consisting of the unbounded functions. Prove that $\mathcal{B}$ and $\mathcal{U}$ are dense subsets of $\mathrm{C}(\mathbb{N}, \mathbb{R})$.

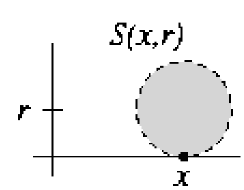
VIII. The figure to the right shows a 2-fold covering map $p\colon (X, x_0) \to (Y, y_0)$ between two 2-dimensional surfaces; the covering transformation is rotation by an angle π around the vertical axis δ shown in the figure. The figure shows a loop in Y that encircles one of its handles, and its two preimage loops in X.



1. Draw a loop α based at y_0 that represents a nontrivial element of the image of $p_\#\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$, and draw its lift $\widetilde{\alpha}$ starting at x_0.
2. Draw a loop β based at y_0 that represents an element not in the image of $p_\#\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$, and draw its lift $\widetilde{\beta}$ starting at x_0.
3. What is the index of $p_\#(\pi_1(X, x_0))$ in $\pi_1(Y, y_0)$? Why?

IX. For $S \subseteq [0, 1]$, define $J(S)$ to be the smallest closed connected subset in $[0, 1]$ that contains S. Let $\mathcal{U} = \{ U_1, \ldots, U_n \}$ be a finite open cover of a normal space X. Suppose that for each U_i, a number $t_i \in [0, 1]$ has been selected. For each $x \in X$, let $S(x) = \{ t_i \mid x \in U_i \} \subset [0, 1]$. Prove that there exists a continuous function $f\colon X \to [0, 1]$ such that for every $x \in X$, $f(x) \in J(S(x))$.

X. Let $X = \{ (x, y) \in \mathbb{R}^2 \mid y \geq 0 \}$. For $x \in \mathbb{R}$ and $r > 0$, let $D(x, r)$ be the open disc in X with center (x, r) and radius r, and put $S(x, r) = D(x, r) \cup \{ (x, 0) \}$. Let $\mathcal{T}$ be the topology on X generated by the subbasis $\{ S(x, r) \mid x \in \mathbb{R},\ r > 0 \}$. Let $A = \{ (a, 0) \mid a \in \mathbb{Q} \}$ and $B = \{ (b, 0) \mid b \in \mathbb{R} - \mathbb{Q} \}$.



1. Prove that A and B are closed subsets of $(X, \mathcal{T})$.
2. Suppose that there exist disjoint open subsets U and V in $(X, \mathcal{T})$ with $A \subseteq U$ and $B \subseteq V$. In $\mathbb{R} - \mathbb{Q}$ (with its usual topology as a subspace of $\mathbb{R}$), let C_n be the closure of $\{ b \in \mathbb{R} - \mathbb{Q} \mid S\left(b, \frac{1}{n}\right) \subset V \}$. Prove that C_n does not contain any open subset of $\mathbb{R} - \mathbb{Q}$.
3. Deduce that X is not a normal space.