Analysis Exam, Spring 2002

Name (please print):

Student No.:

1. State the Lebesgue's Dominated Convergence Theorem.

2. State the Fubini Theorem.

3. State the Riesz Representation Theorem.

4. State Jensen's Inequality.

5. State the Cauchy's Integral Formula for holomorphic functions.

6. State the Maximum Modulus Principle for harmonic functions.

7. Find the Lebesgue measure of the set $E \subseteq \mathbb{R}^2$ of all points (x, y) such that

$$0 \leq x \leq 1,\ 0 \leq y \leq 1,\ sin(x) < \frac{1}{2}, \text{ and } cos(x+y) \text{ is irrational.}$$

8. Let $f \in L^1(\mathbb{R}^n)$. Show that

$$\lim_{\epsilon \longrightarrow 0} \int_{\mathbb{R}^n} |f(x+\epsilon) - f(x)|\, dx = 0.$$

9. Let $f_1, f_2 \in L^1(\mathbb{R}^n)$. Show that for almost every $x \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} |f_1(t) f_2(x-t)|\, dt < \infty.$$

10. Suppose $f_1, f_2 \in L^1(\mathbb{R}^n)$ and f_1 is bounded. Show that the formula

$$f_1 * f_2(x) = \int_{\mathbb{R}^n} f_1(t) f_2(x-t)\, dt$$

defines a continuous function of $x \in \mathbb{R}^n$.

11. Show that for any measurable set $E \subseteq \mathbb{R}^n$ of positive Lebesgue measure, the set

$$E - E = \{x - y;\ x, y \in E\}$$

contains a non-empty open neighbourhood of $0 \in \mathbb{R}^n$.

12. Prove that the set of points where a sequence of measurable functions is convergent is a measurable set.

13. Suppose $f : \mathbb{R} \to [0, \infty)$ is a measurable function. Let

$$A(f) = \{(x, y);\ 0 < y < f(x)\}.$$

Show that the set $A(f)$ is Lebesgue measurable and compute its measure.

14. Suppose μ is a positive measure on a set X, with $\mu(X) < \infty$. Show that for any measurable function $f : X \to \mathbb{C}$,

$$\| f \|_r \leq \| f \|_s \qquad (0 < r < s < \infty).$$

15. Let C be the positively oriented unit circle in the complex plane. Compute the integral

$$\int_C \frac{e^z - e^{-z}}{z^4}\, dz.$$

16. Suppose f and g are entire functions such that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Show that f is a constant multiple of g.

17. Explain how does the Fundamental Theorem of Calculus imply the Fundamental Theorem of Algebra.

18. Let

$$u(z) = Im\left[\left(\frac{1+z}{1-z}\right)^2\right] \qquad (z \in \mathbb{C},\ |z| < 1).$$

Show that u is a harmonic function and that for all $\theta \in \mathbb{R}$,

$$\lim_{r \to 1} u(re^{i\theta}) = 0.$$

Hint: if $|z| = 1$ then $\left(\frac{1+z}{1-z}\right) \in i\mathbb{R}$.