Topology Qualifying Examination

Monday, May 12, 2003, 10:00am–1:00pm.
Answer three questions per section. 9 questions in total.

Section I. **Answer any three questions.**

Q1].. Define what it means for a topological space to be *compact*. Define what it means for a topological space to be *Hausdorff*.

1. Prove that a compact subspace of a Hausdorff space is closed.
2. Prove that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

Q2].. Define *quotient map* and *quotient topology*.

1. Let $\sim$ be an equivalence relation on a topological space X, and let $X/\sim$ denote the set of equivalence classes with the quotient topology. Suppose that $f : X \to Y$ is a continuous, surjective map with the property that $f(x_1) = f(x_2)$ if and only if $x_1 \sim x_2$. Prove that there exists a continuous bijection from $X/\sim$ to Y.
2. Let $\sim$ denote the equivalence relation on $[0, 1]$ defined by $0 \sim 1$. Prove that $[0, 1]/\sim$ is homeomorphic to the circle S^1.

Q3].. True or False (supply a one-line justification for a claim of "True" or a counterexample for a claim of "False")

1. $\mathbb{R}^\omega$ is connected in the product topology.
2. $\mathbb{R}^\omega$ is connected in the box topology.
3. A product of metric spaces is metrizable in the product topology.
4. If the one point compactifications of two locally compact, Hausdorff spaces X and Y are homeomorphic, then X is homeomorphic to Y.

Q4].. True or False (supply a one-line justification for a claim of "True" or a counterexample for a claim of "False")

1. If a space is Lindeloff and regular, then it is normal.
2. If a space is compact and Hausdorff, then it is metrizable.
3. The set $\{(q, -q) \mid q \in \mathbb{Q}\}$ is a closed subspace of the space $\mathbb{R}_l^2$. Here $\mathbb{R}_l$ denotes the set of real numbers with the *lower limit topology*; that is given by the basis $\{[a, b) \mid a < b \in \mathbb{R}\}$
4. Products of regular spaces are regular in the product topology.

Section II. **Answer any three questions.**

Q1].. State the following theorems.

1. The van Kampen Theorem (the more basic version of $X = A \cup B$ is sufficient)
2. The Teitze Extension Theorem
3. The Tychonoff Theorem
4. The (general) lifting criterion for covering spaces [existence of lifts]

Q2].. State the Lebesgue covering lemma for compact metric spaces.

Sketch a proof that if A and B are open in X, A, B and $A \cap B$ are path connected and contain the point x_0, and $X = A \cup B$, then elements of $\pi_1(X, x_0)$ can be expressed as products of elements in $\pi_1(A, x_0)$ and $\pi_1(B, x_0)$.

Q3].. Give the definition of a covering space.

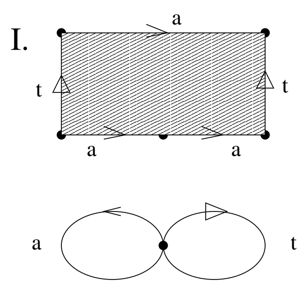
Let $p : \hat{X} \to X$ be a covering space, and let $f, g : Y \to \hat{X}$ be continuous maps which satisfy $p \circ f = p \circ g$ and $f(y_0) = g(y_0)$ for some point $y_0 \in Y$. Prove that if Y is connected, then $f = g$. [uniqueness of lifts]

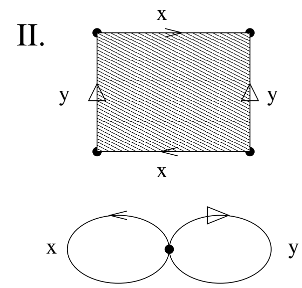
Q4].. This question asks you to explore effects of one point compactifications on the fundamental group. You don't have to give lots of detail in your answers; a one-phrase description of the one point compactifications, together with the two groups is enough in each case.

1. Let $X = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \leq 1\}$ be the punctured disk. What is $\pi_1(X)$? What is π_1 of the one point compactification of X?
2. Let $Y = \mathbb{R}^2 \setminus \{(0, 0)\}$. What is $\pi_1(Y)$? What is π_1 of the one point compactification of Y?
3. Let $Z = \{(x, y, z) \in \mathbb{R}^3 \mid 0 < x^2 + y^2 + z^2 \leq 1\}$ be the punctured 3-dimensional ball. What is $\pi_1(Z)$? What is π_1 of the one point compactification of Z?
4. Let $W = \mathbb{R}^3 \setminus \{(0, 0, 0)\}$. What is $\pi_1(W)$? What is π_1 of the one point compactification of W?
5. Let $M = [0, 1] \times (0, 1)/(0, t) \sim (1, 1 - t)$ be the open Mobius band (without boundary circle). What is $\pi_1(M)$? What is π_1 of the one point compactification of M?

Section III. **Answer any three questions.**

Q1].. This question concerns the following two 2-complexes (each described as the quotient space of the union of a disk and a wedge of two circles).





1. Use van Kampen to determine the fundamental groups of the two 2-complexes.

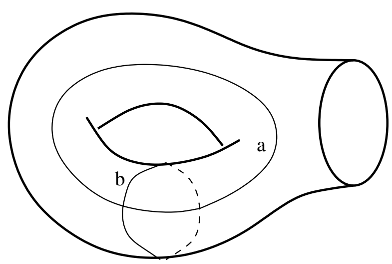
2. Write down the abelianizations of these two groups.

3. Prove that the spaces are not homotopy equivalent.

Q2].. Answer all parts.

1. Define what is meant by a *retraction* from a topological space X to a subset A. [Give a diagram of maps and spaces formulation of the definition]

2. Let X be a torus with an open disk removed as in the diagram. What is $\pi_1(X)$?



3. Express the homotopy class of the boundary circle ∂X as (a conjugacy class of) an element in the group above.

4. Prove that there does not exist a retraction from X to ∂X above.

Q3].. H is the subgroup of the free group on $\{a, b\}$ which is generated by four elements as follows

$$H = \langle a^2,\, b,\, a\bar{b}a,\, a^2b\rangle$$

1. Construct a covering space of the wedge of two circles corresponding to the subgroup H.

2. Is H free? If so, write down a basis for H.

3. What is the index of H in $F_{\{a,b\}}$?

4. Is $ab \in H$? Explain your answer.

Q4].. Answer all parts.

1. Give the definition of an *automorphism* of a covering space $p : \widehat{X} \to X$. [Include a diagram of maps and spaces in your definition]

2. What can you say (no proof necessary!) about an automorphism of a covering space which fixes a point of the covering space?

3. Determine (with simple justifications) the automorphism groups of the following covering spaces of the wedge of two circles.