Topology Qualifying Examination

Monday, August 18, 2003, 10:00am–1:00pm.
Answer three questions per section. 9 questions in total.

Section I. **Answer any three questions.**

Q1].. Define what it means for a topological space to be:

- *compact*
- *Hausdorff*
- *locally compact.*

For the next two parts, let X be a locally compact, Hausdorff topological space.

1. Define the topology on the one-point-compactification $Y = X \cup \{\infty\}$ of X. [You do **not** have to prove that it is a topology]
2. Give a proof that the space Y above is Hausdorff, indicating clearly what hypotheses about the space X are used in your proof.

Q2].. Define what it means for a topological space to be *connected.*

Define the *product topology* on a product $X \times Y$ of topological spaces X and Y.

1. Prove that the continuous image of a connected space is connected.
2. We know that if $\{X_\alpha\}_{\alpha \in J}$ is a family of connected topological spaces, then $\prod_{\alpha \in J} X_\alpha$ is connected in the product topology. What about the converse; if $\prod_{\alpha \in J} X_\alpha$ is connected, does it follow that the X_α are connected? Give a proof or a counterexample.
3. Let $\{X_\alpha\}_{\alpha \in J}$ be a family of connected metric spaces, and give the product $Y = \prod_{\alpha \in J} X_\alpha$ the uniform topology. Is Y connected? Give a proof or counterexample.

Q3].. Define *quotient map* and *quotient topology.*

Let $\sim$ denote the equivalence relation on the cylinder $S^1 \times [-1,1]$ defined by $(\mathbf{v}, -1) \sim (\mathbf{v}', -1)$ for all $\mathbf{v}, \mathbf{v}' \in S^1$, and $(\mathbf{v}, 1) \sim (\mathbf{v}', 1)$ for all $\mathbf{v}, \mathbf{v}' \in S^1$. Here $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ denotes the unit circle, and we are using vector notation $\mathbf{v}$ to denote an element (x,y) of S^1. Prove that the quotient space $S^1 \times [-1,1]/\sim$ is homeomorphic to the unit sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Please state carefully any results about the existence of homeomorphisms, results about the quotient topology, etc that you use in your proof.

Q4].. True or False (Supply a short justification or an appeal to a theorem for a claim of "True". Supply a short justification or a counterexample for a claim of "False".)

1. The set $A = \{(0,y) \mid y \in \mathbb{R}\} \cup \{(1,y) \mid y \in \mathbb{R}\}$ is a retract of $\mathbb{R}^2$.
2. The logarithmic spiral $L = \{(0,0)\} \cup \{(e^t \cos t, e^t \sin t) \mid t \in \mathbb{R}\}$ is a retract of $\mathbb{R}^2$.
3. If X is a second countable, compact, Hausdorff space, then X is metrizable.
4. If the topological space $(X, \mathcal{T}_1)$ is compact, and $\mathcal{T}_2$ is finer than $\mathcal{T}_1$ (that is $\mathcal{T}_1 \subset \mathcal{T}_2$), then $(X, \mathcal{T}_2)$ is also compact.

Section II. **Answer any three questions.**

Q1].. Define what it means for a space to be *simply connected.*

True/False (Supply a short justification or an appeal to a theorem for a claim of 'True". Supply a short justification or a counterexample for a claim of 'False".)

1. If X is a simply connected, locally compact Hausdorff space, then the one point compactification of X is also simply connected.

2. quotient spaces of simply connected spaces are also simply connected.

3. path-connected subspaces of simply connected spaces are also simply connected.

4. the product of two simply connected spaces is also simply connected.

Q2].. State the Lebesgue covering lemma for compact metric spaces.

Suppose that $X = A \cup B$, where A and B are open in X, and that $A \cap B$ is path connected and contains a point x_0 of X. Sketch a proof of the fact that if A and B are simply connected then X is also simply connected.

Suppose that A and B above are just path-connected (not necessarily simply connected). State the van Kampen theorem (no proof necessary), which tells one how to obtain $\pi_1(X, x_0)$ from $\pi_1(A, x_0)$ and $\pi_1(B, x_0)$.

Q3].. Give the definition of a covering space.

Let $p : \hat{X} \to X$ be a covering space, and let $f, g : Y \to \hat{X}$ be continuous maps which satisfy $p \circ f = p \circ g$ and $f(y_0) = g(y_0)$ for some point $y_0 \in Y$. Prove that if Y is connected, then $f = g$. [uniqueness of lifts]

Q4].. In this question $p : \mathbb{R}^2 \to T^2$ denotes the covering map $(x, y) \mapsto (e^{2\pi ix}, e^{2\pi iy})$, and S^2 denotes the 2-sphere $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. All "maps" are continuous.

1. Prove that any map $f : S^2 \to \mathbb{R}^2$ is homotopy equivalent to a constant map.

2. Say why every map $g : S^2 \to T^2$ lifts to a map $S^2 \to \mathbb{R}^2$.

3. Prove that every map $g : S^2 \to T^2$ is homotopy equivalent to a constant map.

Section III. **Answer any three questions.**

Q1].. Let $f : X \to Y$ be a continuous map of topological spaces which takes $x_0 \in X$ to $y_0 \in Y$. Define the *induced homomorphism* $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and prove that it is indeed a group homomorphism.

True/False (Supply a short justification or an appeal to a theorem for a claim of 'True". Supply a short justification or a counterexample for a claim of "False".)

1. If f is injective, then f_* is also injective.
2. If f is surjective, then f_* is also surjective.
3. If f is a homeomorphism, then f_* is an isomorphism.

Q2].. Retractions.

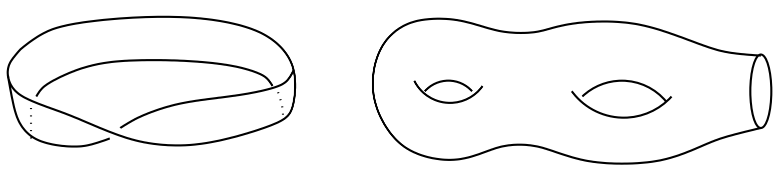
1. Define what is meant by a *retraction* from a topological space X to a subset A. [Give a diagram of maps and spaces formulation of the definition]
2. If r is a retraction, prove that r_* is surjective.

Now attempt **just one** of the following.

- Prove that there does not exist a retraction from the once punctured genus two surface to its boundary circle.

  OR

- Prove that there does not exist a retraction from the Mobius band to its boundary circle.

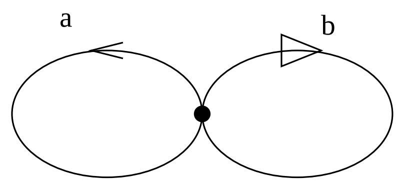


Q3].. H is the subgroup of the free group on $\{a, b\}$ which is generated by four elements as follows

$$H = \langle a^2,\, b^2,\, ab^2a,\, a\bar{b} \rangle$$

1. Construct a covering space of the wedge of two circles corresponding to the subgroup H.
2. Is H free? If so, write down a free basis for H.
3. What is the index of H in $F_{\{a,b\}}$? Explain why.
4. IS H normal in $F_{\{a,b\}}$? Explain why/why not.
5. Is $aba \in H$? Explain why/why not.

Q4].. Write down all distinct (up to isomorphism) connected, 3-fold covering spaces of the wedge of two circles below.



Define the notion of a *regular covering space*, and say which of your covers above are regular and which are not.

Define the notion of an *automorphism* of a covering space, and write down the automorphism groups of your covering spaces above (give brief justifications for your claims).