Topology MA and Qualifying Exam
May, 2004

Instructions. Give clear and complete arguments for each of the problems, but try to avoid excessive detail. You may use major theorems when appropriate but clarify their use by either naming or stating the theorem. Unless stated otherwise X, Y and Z denote topological spaces.

Part I. *Work six of the problems in this section.*

1. Let $\mathcal{B}$ be the family of subsets of the real line $\mathbb{R}$ consisting of the singleton set $\{0\}$ and all of the open intervals (a, b) where $a < b$ and either a or b is rational.
(a) Show that $\mathcal{B}$ is the basis for some topology $\mathcal{T}$ on $\mathbb{R}$.
(b) Show that the topology $\mathcal{T}$ is finer than the Euclidean topology.
(c) Show that the topology $\mathcal{T}$ is strictly finer than the Euclidean topology.

2. (a) Describe the open sets in the product topology on $\prod_{k=1}^{\infty} \mathbb{R}$ (where each factor $\mathbb{R}$ has the Euclidean topology).
(b) Let $X_n = (x_{n,k})_{k=1}^{\infty}$ where $x_{n,k} = 0$ if $k \neq n$ and $x_{n,k} = 1$ if $k = n$. Show that the sequence (X_n) converges in $\prod_{k=1}^{\infty} \mathbb{R}$ (with product topology).

3. Let $f : X \to Y$ be a continuous surjection. Either prove or disprove each of the following:
(a) If Y is compact then X is compact.
(b) If X is compact then Y is compact.

4. (a) Suppose X contains an infinite closed subspace A which has the discrete topology. Show that X is neither compact nor sequentially compact.
(b) Show that if $p : \widetilde{X} \to X$ is an infinite-sheeted covering map where X is Hausdorff then $\widetilde{X}$ cannot be compact.

5. Let X be homotopy equivalent to a singleton space.
(a) Prove that X is path connected.
(b) Prove that X is simply connected.

6. Let X be a topological space and $A \subset X$.
(a) Show that if x is a limit point of A then $x \in \overline{A}$.
(b) Show that if $x \in \overline{A}$ then x need not be a limit point of A.

7. Let $f : X \to Y$ be continuous surjection, and let $g : Y \to Z$ be a function.
(a) If f is a closed map and $g \circ f$ is continuous then show that g is continuous.
(b) Show that the conclusion in part (a) may fail if f is not assumed to be a closed map.

Part II. *Work at least two of the problems from this section. Give proofs which rely predominantly on basic definitions in this part.*

8. Show that a connected, locally path connected space is path connected, but that there are connected spaces which are not path connected.

9. Show that a compact Hausdorff space is regular, but that the conclusion may be false if either of the hypotheses are removed.

10. Prove that a metrizable space is second countable if and only if it is separable. Show that in general second countability and separability are distinct topological properties.

Part III. *Work at least two of the problems from this section. When possible, give proofs which stem from basic definitions.*

11. Let X_n be the space obtained from the unit disk $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$ in the Euclidean plane by removing n disjoint open 2-disks from the interior of D, where n is a nonnegative integer (that is, $n \geq 0$). Let $S = \{(x, y) \mid x^2 + y^2 = 1\} \subset X_n$
(a) For which n is S a deformation retract of X_n?
(b) For which n is S a retract of X_n?

12. (a) Let $X = \cup_{k=1}^{\infty} X_k$ where X_k is a simply connected open subspace of X and $X_k \subset X_{k+1}$ for each positive integer k. Show that X is simply connected.
(b) Let A be a simply connected space with $x, y \in A$. Let X be the quotient space obtained from $A \cup [0, 1]$ by identifying x with 0 and y with 1. Construct the universal covering map of X.
(c) Use (b) to show that the fundamental group of X is infinite cyclic.

13. (a) Describe the compact-open topology on the function space $Y^Z = \{\text{functions from } Z \text{ to } Y\}$.
(b) Let $f : X \to Y$ be a continuous function. Define $F : X^Z \to Y^Z$ by $F(g) = f \circ g$ for each function g from Z to X. Show that F is continuous with respect to the compact-open topology.