Topology MA and Qualifying Exam
August 18, 2004

Instructions. Give clear and complete arguments for each of the problems, but try to avoid excessive detail. You may use major theorems when appropriate but clarify their use by either naming or stating the theorem. Unless stated otherwise X and Y denote topological spaces.

Part I. *Work six of the problems in this section.*

1. Consider $\mathbb{R}^2$ with the Euclidean metric. For each $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and each $\epsilon > 0$ let $C_\epsilon(\mathbf{x})$ be the union of the two ϵ balls centered at (x_1, x_2) and at $(x_1 + \epsilon, x_2)$. Show that the collection of sets $\{C_\epsilon(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^2 \text{ and } \epsilon > 0\}$ forms a basis for the Euclidean topology on $\mathbb{R}^2$.

2. (a) Let X be a compact space. Show that a subset of X with no limit points must be finite.
(b) Let X be compact and first countable. Show that every sequence in X has a convergent subsequence.

3. Give a three or four sentence summary of a proof of the Intermediate Value Property: *Let X be connected and let $f : X \to \mathbb{R}$ be continuous. If $a, b \in X$ and r is a real number between $f(a)$ and $f(b)$ then $r = f(c)$ for some $c \in X$.*

4. Using the definition of the set $\pi_1(X, x_0)$ and the definition of the group operation on this set, show that $\pi_1(X, x_0)$ satisfies the identity element axiom for a group. (*There exists* $1 \in \pi_1(X, x_0)$ *such that* $1 \cdot g = g = g \cdot 1$ *for each* $g \in \pi_1(X, x_0)$.)

5. (a) Using only the definition of compactness, show that every compact subset of a Hausdorff space is closed.
(b) Give an example which illustrates that Hausdorffness is necessary in part (a).

6. (a) Define the two terms: 'homotopy between two functions' and 'homotopy equivalence between two topological spaces'.
(b) Give an example of topological spaces X and Y which have the same homotopy type but are not homeomorphic.
(c) Give an example of topological spaces X and Y which have isomorphic fundamental groups but do not have the same homotopy type.

7. (a) When we say that 'local compactness is a topological property' what does that mean?
(b) Show that local compactness is a topological property.

Part II. *Work at least three of the problems from this section.*

8. Show that every second countable space is separable, then give an example showing that the converse of this statement is false.

9. Let S be the closed orientable surface with genus two and let U and V be open subsets of S as shown in the figure below. Apply Van Kampen"s Theorem to this decomposition to determine a presentation for $\pi_1(S, x_0)$ where x_0 is a point in $U \cap V$. (Hint: U (as well as V) is a punctured torus. Start by describing a deformation retract of U which is a graph.)

10. Let Y be an interval in the real line with the Euclidean topology. Show that Y is connected.

11. (a) Give an example (by picture) of a covering space of degree four over the 'theta graph' X.
(b) Describe as many different (up to equivalence) covering spaces of degree four over the theta graph as you can. How many are there altogether?

12. Let X be a path connected space which is the union of two path connected open subsets U and V. Let x_0 be an element of $U \cap V$.
(a) If U is simply connected and $U \cap V$ is path connected show that the inclusion map from V to X determines an isomorphism from $\pi_1(V, x_0)$ to $\pi_1(X, x_0)$.
(b) Suppose that all of the conditions in (a) are met except that $U \cap V$ is path connected. Show that the conclusion of (a) may fail.