Algebra qualifying exam

May 2005

1. Prove that if the quotient group $G/Z(G)$ of the group G by its center $Z(G)$ is cyclic, then G is abelian.

2. Let G be a non-cyclic group of order 21. How many 3-Sylow subgroups does G have? Explain your answer.

3. Let V be a finite dimensional vector space over $\mathbb{C}$, and $T : V \to V$ be a linear operator. Suppose that for some positive integer n, $T^n(v) = 0$ for all $v \in V$. Prove that there exists a basis in V, such that the matrix of T with respect to this basis is strictly upper triangular, i.e. ij-th entry is zero when $i \geq j$.

4. Let $f = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ be a monic polynomial with integer coefficients.

   a) If f has a rational root, show that this root is an integer.

   b) Let p be prime, and $\bar{a}$ denote the image of $a \in \mathbb{Z}$ under the canonical homomorphism $\mathbb{Z} \to \mathbb{Z}_p$. Let $\bar{f} = \sum \bar{a}_i x^i \in \mathbb{Z}_p[x]$. If $\bar{f}$ is irreducible in $\mathbb{Z}_p[x]$ for some prime p, prove that f is irreducible in $Z[x]$

5. Let R be a commutative ring and M be an R-module. Prove that M is cyclic if and only if $M \cong R/I$ for some ideal I of R.

6. Consider the subring R of $\mathbb{Q}[x]$ of polynomials with integer constant term
$$R = \{a_0 + a_1 x + ... + a_n x^n \in \mathbb{Q}[x] | a_0 \in \mathbb{Z}\}.$$

   a) Prove that R is an integral domain.

   b) Which of the ideals $\{f(x) \in R | f(0) = 0\}$, (x), (p), $p \in \mathbb{Z}$ prime, are prime? maximal? Explain.

   c) Is R a principal ideal domain? If yes, prove it. If not, give an example of an ideal of R which is not principal.

7. Let $\mathbb{F}$ be a finite field.

   a) Show that the characteristic of $\mathbb{F}$ is $p > 0$ for some prime number p.

   b) Show that the cardinality of $\mathbb{F}$ is $q = p^n$, for some integer $n \geq 1$ and every element $x \in \mathbb{F}$ satisfies $x^q - x = 0$

   c) Show that any two finite fields of the same cardinality are isomorphic.

8. Let the field F be an extension of a field K. If $u, v \in F$ are algebraic over K of degrees m and n respectively, show that $[K(u,v) : K] \leq mn$. If $(m,n) = 1$, show that $[K(u,v) : K] = mn$.