In the following m denotes the Lebesque measure on $\mathbb{R}$. If necessary we specify with m_x the variable with respect to which the integral is taken.

1) Let $f:[a,b]\to\mathbb{R}$ show that the function g defined by
$$g(y)=\sup_{\delta>0}\inf_{|x-y|<\delta} f(x)\ (=\sup\{\inf\{f(x)\mid |x-y|<\delta\}\mid \delta>0\})$$
is lower semicontinuous.

2) Let (f_n) be a sequence of measurable functions defined on $\mathbb{R}$ such that $f_n\to f$ a.e. and $\int_{\mathbb{R}}|f_n|dm\to\int_{\mathbb{R}}|f|dm<\infty$
   a) Show that for each measurable set E we have
   $$\int_E f_n dm\to\int_E f dm$$
   b) Is the statement still true if we require only $\int_{\mathbb{R}} f_n dm\to\int_{\mathbb{R}} f dm$. If not, provide a counterexample.

3) Let E be a measurable set in $\mathbb{R}$ with $0<m(E)<\infty$
   a) Show that the function $\phi\colon\mathbb{R}\to\mathbb{R}$ given by
   $$\phi(x)=\int_{\mathbb{R}}\chi_E(y)\chi_E(x+y)dm_y$$
   is continuous.
   b) Show that the set $F=\{x-y\in\mathbb{R}\mid x,y\in E\}$ contains open neighborhoods of all $x\in\mathbb{R}$ for which $\phi(x)\neq 0$.

4) Let $f\colon\mathbb{R}\to\mathbb{R}$ be integrable show that we have
$$\int_{(y,y+1)} f dm\to 0 \text{ as } y\to\infty.$$

5) Let $(X,\mathcal{B})$ be a measurable space. Let μ and ν be to measures on $\mathcal{B}$, with $\mu>\nu$
   a) Show that there is a measure λ on $\mathcal{B}$ such that $\mu=\lambda+\nu$.
   b) If ν is σ-finite show that λ is unique.
   c) Show that there is always a smallest such measure λ, but that in general λ is not unique.

6) Let $(X,\mathcal{B},\mu)$ be a complete measure space and let (f_n) be a sequence of measurable functions converging to f a.e.
   a) Provide three different, additional sets of assumptions which would allow you to conclude
   $$\lim_{n\to\infty}\int_X f_n d\mu=\int_X f d\mu.$$

b) Show that the Monotone Convergence Theorem implies the Lemma of Fatou.

7) Let $(X, \mathcal{B})$ be a measurable space and let μ and ν be two measures on $\mathcal{B}$. The measure ν is called μ-continuous if for all $\epsilon > 0$ there is an δ such that

$$(\mu(E) < \delta) \Rightarrow (\nu(E) < \epsilon).$$

If ν is finite show that ν is μ-continuous if and only if ν is absolutely continuous with respect to μ.

8) Show that a countable subset of $\mathbb{R}$ has Hausdorff dimension zero.

9) (Work either a) or b)!)
a) Show that $\log|x|$ is not in $W^{1,1}((1,1))$.
b) Let μ be an invariant measure on the homogeneous space (X, G) and f a μ-integrable function on X. Show that for all $g \in G$ we have

$$\int_X f \circ g d\mu = \int_X f d\mu.$$