Algebra qualifying exam

August 2005

1. Let G be a p-group (i.e. $|G| = p^n$, p-prime, $n \in \mathbb{N}$) and Z be the center of G. Prove that $Z \neq \{e\}$.

2. Let G be a finite group. Suppose that G has a normal subgroup N whose order $|N|$ and index $[G : N]$ are relatively prime. Show that N is the only subgroup of G of order $|N|$.

3. Let L be a linear operator on a finite dimensional vector space V. Prove that there exists a decomposition $V = U \oplus W$ where each summand is L-invariant, $L|_U$ is nilpotent and $L|_W$ is non-singular.

4. If p is a prime number, prove that the polynomial $x^n - p$ is irreducible over the rational numbers.

5. Let M be a (left) module over a (commutative) principal ideal domain $\mathcal{A}$, and for each $m \in M$ let

$$I_m = \{a \in \mathcal{A} | am = 0\}.$$

   Prove

   a) I_m is an ideal of $\mathcal{A}$.

   b) $M_t = \{m \in M | I_m \neq 0\}$ is a submodule of M.

   c) If $p \in \mathcal{A}$ is prime and $p^i m = 0$, then $I_m = (p^j)$, with $0 \leq j \leq i$.

6. In $\mathbb{Z}[x]$, let $I = \{f(x) \in \mathbb{Z}[x] | f(0) = 2m, m \in \mathbb{Z}\}$. Is I a prime ideal of $\mathbb{Z}[x]$? Is I a maximal ideal?

7. Let F be a finite field of characteristic p. Suppose $F \subset F(\alpha)$, where $\alpha^p \in F$. Show that $[F(\alpha) : F]$ is equal to 1 or p.

8. Suppose that K is an extension field of F and $a \in K$ is algebraic. If the degree of the minimal polynomial of a is odd, show that $F(a^2) = F(a)$.