Algebra Qualifying Exam - May 2006

Full marks for complete answers to five *questions.*
Show all work fully *and* clearly.
Good luck!

1. (i) Let G be a group and let Z denote the center of G. Prove that G/Z cannot be a nontrivial cyclic group.

   (ii) Let G be a non-abelian group of order p^3 where p is a prime. Prove that the center of G has order p and coincides with the commutator subgroup (or derived group) of G.

   (iii) With G as in (ii), show that a conjugacy class of G contains one element or p elements. Deduce that G has $p^2 + p - 1$ conjugacy classes.

2. (i) State the Sylow theorems.

   (ii) Let G be a group of order p^2q where p and q are distinct primes. Prove that G is not simple.

   (iii) Let K be a normal subgroup of a finite group G and let P be a Sylow p-subgroup of K (for some prime p). Prove that $G = KN_G(P)$.

3. (i) Let G be a non-abelian group of order 6. Prove that G is isomorphic to S_3.

   (ii) Prove that A_4 has no subgroup of order 6.

   (iii) Determine all natural numbers n for which the alternating group A_n is nilpotent.

4. (i) Give an example, with proof, of a nonzero prime ideal in $\mathbb{Z}[x]$ that is not maximal.

   (ii) Let $S = \{1, 2, \ldots, n\}$ and let $\mathfrak{R}$ denote the ring of all functions $f : S \to \mathbb{C}$ with pointwise operations.

   (a) Prove that $\mathfrak{R} \cong \mathbb{C}^n$ (as rings).

   (b) Describe the maximal ideals of $\mathfrak{R}$.

5. (i) Let F be a finite field and let n be a positive integer. By using the theory of finite fields, or otherwise, prove that $F[x]$ contains an irreducible polynomial of degree n.

   (ii) Suppose now that R is a finite commutative ring with just two maximal ideals. Use the Chinese remainder theorem and (i) to show that, for any integer $n > 1$, there is a monic polynomial $f(x) \in R[x]$ of degree n such that $f(r)$ is a unit in R for every $r \in R$.

6. Prove that the polynomial $x^4 + 1$ is irreducible over $\mathbb{Q}$ but is reducible over $\mathbb{F}_p$, the finite field with p elements, for all primes p.

7. (i) Let L/K be a Galois extension of fields such that $\mathrm{Gal}\,(L/K)$ is non-abelian of order 6. Determine the number of cubic intermediate fields (i.e., the number of fields L_1 such that $K \subset L_1 \subset L$ and $[L_1 : K] = 3$).

   (ii) Give an example of a tower of fields $K \subset L \subset M$ in which L/K and M/L are both Galois extensions but M/K is not Galois.

8. Let K be a field with $\mathrm{char}\, K \neq 2$. We say that K is *quadratically closed* if $K^\times = (K^\times)^2$ (i.e., every element of $K^\times$ is a square). Suppose that K and every finite extension of K is quadratically closed. If L/K is a (finite) Galois extension show that $[L : K]$ is odd. What if L/K is separable but not Galois?

9. (i) Prove that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

   (ii) Let R be a commutative ring and let I be an ideal in R. Prove that $\mathrm{ann}(R/I) = I$. (Recall that, for M an R-module, $\mathrm{ann}(M)$ denotes the set of all $r \in R$ such that $rm = 0$ for all $m \in M$.)

   (iii) Let R be a commutative ring such that every nonzero R-module is free. Prove that R is a field.

10. Let n be a positive integer. Determine the number of conjugacy (or similarity) classes of elements $A \in M_n(\mathbb{C})$ such that $A^2 = 0$.