Algebra Qualifying Exam - August 2006

Full marks for complete answers to five *questions.*
Show all work fully *and* clearly.
Good luck!

1. (i) Let G be a group and let Z denote the center of G. Prove that G/Z is never a nontrivial cyclic group.

   (ii) For any group G, the set $\operatorname{Aut} G$ of all automorphisms of G is a group under composition. Prove that if $\operatorname{Aut} G$ is cyclic then G must be abelian.

2. (i) Prove that there is no simple group of order $160 = 2^5 \cdot 5$.

   (ii) Prove that there is no simple group of order pqr where p, q, r are distinct primes.

3. (i) Determine the number of Sylow 2-subgroups of the symmetric group S_4.

   (ii) Determine the number of Sylow 2-subgroups of the alternating group A_4.

   (iii) Determine all natural numbers n for which the alternating group A_n is solvable.

4. (i) Let R be a commutative ring in which every ideal is prime. Prove that R is a field. (Hint: For $x \neq 0$, consider (x) and (x^2).)

   (ii) Let F be a field and let R denote the subring of $F[x]$ consisting of all polynomials with linear term zero, i.e., $R = F \oplus x^2 F[x]$.

   (a) Show that each nonzero nonunit in R is a product of irreducible elements in R.

   (b) Prove that R is not a UFD. (Hint: Show that x^2 and x^3 are irreducible in R.)

5. Prove that the polynomial $x^4 + 1$ is irreducible over $\mathbb{Q}$ but is reducible over $\mathbb{F}_p$, the finite field with p elements, for all primes p.

6. (i) Determine all primes p for which $x^2 + 1$ is irreducible in $\mathbb{F}_p[x]$.

   (ii) Show that, for any prime p, the polynomial $f(x) = x^p - x - 1$ is irreducible in $\mathbb{F}_p[x]$. (Hint: Look at the roots of $f(x)$ in a splitting field.)

7. (i) Let L/K be a finite Galois extension of fields with Galois group G. Let H be a subgroup of G. Show that there is an $\alpha \in L$ such that $H = \{\sigma \in G : \sigma(\alpha) = \alpha\}$.

   (ii) Give an example, with proof, of a finite extension of fields that is not separable.

8. (i) Show that $\mathbb{Q}(\sqrt{-1}, \sqrt[4]{2})/\mathbb{Q}$ is a Galois extension and determine its Galois group.

   (ii) Show that $\mathbb{Q}(\sqrt{3}, \sqrt{5})/\mathbb{Q}$ is a Galois extension and determine its Galois group.

9. (i) Prove that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

   (ii) Let R be a commutative ring and let I, J be ideals of R such that R/I is isomorphic to R/J (as R-modules). Prove that $I = J$.

10. (i) Determine the number of conjugacy (or similarity) classes of elements $A \in M_6(\mathbb{C})$ such that $A^3 = 0$.

    (ii) Suppose that $A \in M_2(\mathbb{Q})$ satisfies $A^5 = I$ (where I denotes the 2×2 identity matrix). Prove that $A = I$.