Department of Mathematics
M.A. Comprehensive/Ph.D. Qualifying Exam in Analysis
Summer 2006

1. For any subset E of $\mathbf{R}$ and any $k > 0$, define $kE = \{y \in \mathbf{R} : y = kx \text{ for some } x \in E\}$.
(a) Show that $|kE|_e = k|E|_e$. (Here $|E|_e$ denotes the Lebesgue outer measure of E.)
(b) Show that E is measurable if and only if kE is measurable.

2. *(a)* State Fatou's Lemma.

(b) Suppose $\{f_n\}$ is a sequence of measurable functions in L^p, with $\|f_n\|_p \leq 1$ for all n, and suppose $\{f_n(x)\}$ converges to $f(x)$ for almost every $x \in \mathbf{R}$. Prove that $f \in L^p$ and $\|f\|_p \leq 1$.

(c) Show that the following statement is FALSE:
"If $\{f_n\}$ is a sequence in $L^1(\mathbf{R})$, then $\int_{-\infty}^{\infty}(\liminf f_n) \leq \liminf(\int_{-\infty}^{\infty} f_n)$."
Hint: consider

$$f_n(x) = \begin{cases} -1 & \text{for } x \in [n, n+1] \\ 0 & \text{for } x \notin [n, n+1]. \end{cases}$$

3. Suppose $\{f_n\}$ is a sequence of measurable functions which converges in measure to f on a measurable set E, and $\{g_n\}$ is a sequence of measurable functions which converges in measure to g on E. Show that if $h_n = f_n + g_n$, then $\{h_n\}$ converges in measure to $f + g$ on E.

4. Suppose f is measurable on $\mathbf{R}$ and $|f(x)| < \infty$ for almost every $x \in \mathbf{R}$. Prove that there exists a measurable subset E of $\mathbf{R}$ such that $|E| > 0$ and f is bounded on E. (Here $|E|$ denotes the Lebesgue measure of E.)

5. Suppose f is absolutely continuous on $\mathbf{R}$.

(a) Show that if $Z \subseteq \mathbf{R}$ and $|Z| = 0$, then $|f(Z)| = 0$.

(b) Show that if $E \subseteq \mathbf{R}$ and E is measurable, then $f(E)$ is measurable.
Hint: first show that if f is continuous and E is open (or closed), then $f(E)$ is measurable.

6. Suppose f is measurable on $\mathbf{R}$ and $\int_{-\infty}^{\infty} |tf(t)|\, dt < \infty$. Prove that for almost every $x > 0$,

$$\frac{d}{dx}\int_{-\infty}^{\infty} \sin(xt)f(t)\, dt = \int_{-\infty}^{\infty} t\cos(xt)f(t)\, dt.$$

Justify your calculations carefully, using theorems about the Lebesgue integral.
Hint: Start with the fact that $\sin(xt) = t\int_0^x \cos(st)\, ds$.

7. Suppose $\{f_n\}$ and $\{g_n\}$ are sequences in L^2, and f and g are functions in L^2, such that:

(i) $\lim \|f_n - f\| = 0$,
(ii) there exists $M > 0$ such that $\|g_n\| \leq M$ for all $n \in \mathbf{N}$, and
(iii) $\lim\langle g_n, h\rangle = \langle g, h\rangle$ for all $h \in L^2$.

Prove that $\lim\langle f_n, g_n\rangle = \langle f, g\rangle$.

8. Suppose S is a set with infinitely many elements, and let $\mathcal{A}$ be the collection of all subsets E of S such that either E or $S - E$ is finite.
(a) Show that if E_1 and E_2 are in $\mathcal{A}$, then $E_1 \cup E_2$ is in $\mathcal{A}$.
(b) Show that if E is in $\mathcal{A}$, then $S - E$ is in $\mathcal{A}$.
(c) Show that $\mathcal{A}$ is not a σ-algebra.