Algebra Qualifying Exam

May 2007, University of Oklahoma

1. Let $\zeta = e^{2\pi i/8}$. Analyze the representation $\mathbb{Q}(\zeta)/\mathbb{Q}$: Show that it is a Galois extension, find the degree, find the minimal polynomial of ζ over $\mathbb{Q}$, determine the Galois group, and describe all intermediate fields explicitly.

2. Let E/F be a Galois extension of degree 99. Show that there is a unique intermediate field M of degree 11 over F, and that M is Galois over F.

3. For a prime number p let $\mathbb{F}_{p^n}$ be the field with p^n elements.

a) List all intermediate fields of the extension $\mathbb{F}_{p^{10}}/\mathbb{F}_p$.

b) Show that $\mathbb{F}_{p^{10}}$ contains exactly $p(p^9 - p^4 - p + 1)$ elements α such that $\mathbb{F}_{p^{10}} = \mathbb{F}_p(\alpha)$.

c) Determine the number of monic, irreducible polynomials of degree 10 with coefficients in $\mathbb{F}_p$.

4. Consider the polynomial ring $\mathbb{Z}[X]$.

a) Let $I = \{a_0 + a_1X + \ldots + a_nX^n \in \mathbb{Z}[X] : \ a_0 + \ldots + a_n = 0\}$. Show that I is an ideal. Is it a prime ideal? A maximal ideal?

b) For a prime number p let $J = \{a_0 + a_1X + \ldots + a_nX^n \in \mathbb{Z}[X] : \ a_0 + \ldots + a_n \in p\mathbb{Z}\}$. Show that J is an ideal. Is it a prime ideal? A maximal ideal?

5. Let G be an abelian group, and let $H \subset G$ be the subset of all elements of finite order.

a) Show that H is a subgroup of G.

b) Show that every element of G/H except the identity element has infinite order.

c) In the case $G = \{z \in \mathbb{C}^\times : \ |z| = 1\}$, show that H is isomorphic to the additive group $\mathbb{Q}/\mathbb{Z}$.

6. A matrix $M \in M(n \times n, \mathbb{C})$ is called *idempotent* if $M^2 = M$. Let S be the set of all idempotent matrices in $M(n \times n, \mathbb{C})$.

a) Show that the group $G = \mathrm{GL}(n, \mathbb{C})$ acts on S via conjugation.

b) Determine the number of orbits for this action.

7. Let $\mathbb{F}_p$ be the field with p elements.

a) Determine the number of elements of $\mathrm{GL}(2, \mathbb{F}_p)$.

b) Determine the number of elements of $\mathrm{SL}(2, \mathbb{F}_p)$.

c) Find a Sylow p–subgroup of $\mathrm{GL}(2, \mathbb{F}_p)$.

8. Let G be a finite group and $P < G$ a Sylow p-subgroup. Let $N_G(P)$ be the normalizer of P in G. Let $H < G$ be a subgroup containing $N_G(P)$. Prove that $N_G(H) = H$.