Qualifying Exam: Analysis

Name:...

1. **(3+3 points)** Let $E \subset X$, $E \neq \emptyset, X$.
(a) Find the σ-algebra $\mathcal{M} = \mathcal{M}(\mathcal{F})$ that is generated by $\mathcal{F} = \{E\}$.
(b) What functions $f : X \to \mathbb{C}$ are measurable if we use this σ-algebra on X (and the Borel algebra on $\mathbb{C}$)?

2. **(5 points)** Let μ, ν be finite Borel measures on $\mathbb{R}$ and assume that
$$\mu((a,\infty)) = \nu((a,\infty)) \qquad \text{for all } a \in \mathbb{R}.$$
Show that then $\mu(B) = \nu(B)$ for all Borel sets $B \subset \mathbb{R}$.
Suggestion: Use the regularity of these measures and the fact that open subsets of $\mathbb{R}$ are countable disjoint unions of open intervals.

3. **(3 points)** Let μ be a measure on $(X, \mathcal{M})$ and let $f : X \to [0,\infty]$ be a non-negative measurable function. Prove that then
$$\nu(E) = \int_E f(x)\, d\mu(x)$$
defines a new measure on $(X, \mathcal{M})$.

4. **(3 points)** Let $f \in L^1(\mathbb{R})$. Show that then
$$g(t) = \int_{-\infty}^{\infty} f(x) \sin xt\, dx$$
is a continuous function on $\mathbb{R}$.

5. **(7 points)** Evaluate
$$\int_0^\infty dx \int_1^\infty dy\, e^{-(1+i)xy^2}.$$
You will probably apply the Fubini-Tonelli Theorem here; please justify this carefully (don't just give the formal calculation).

6. **(5 points)** Let ν be the Borel measure on $\mathbb{R}$ that is generated by the increasing, right-continuous function
$$F(x) = \begin{cases} 0 & x < 0 \\ 1 + 2x & x \geq 0 \end{cases}$$
(so $\nu((-\infty, x]) = F(x)$). Find the Lebesgue decomposition of ν with respect to Lebesgue measure $\mu = m$, and determine the Radon-Nikodym derivative of the absolutely continuous part of ν.

7. **(2+2+3+4 points)** For what p ($1 \le p \le \infty$) are the following functions in $L^p(0,\infty)$:

$$\text{(a) } f(x) = \frac{x}{x+1}; \quad \text{(b) } f(x) = \frac{1}{(x+1)^{1/2}};$$

$$\text{(c) } f(x) = \frac{e^{-x}}{x^{1/2}}; \quad \text{(d) } f(x) = \sum_{n=1}^{\infty} n\chi_{(n,n+2^{-n})}(x)$$

8. **(4 points)** Let $F : \mathbb{R} \to \mathbb{C}$ be absolutely continuous with $F' \in L^p(\mathbb{R})$, $1 \le p < \infty$. Show that there exists a constant $C > 0$ so that

$$|F(x) - F(y)| \le C|x-y|^\alpha \qquad (x, y \in \mathbb{R}),$$

with $\alpha = 1 - 1/p$.

9. **(3+4 points)** (a) Let $x_j \in [0,1]$, and suppose that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} f(x_j) \text{ exists for all } f \in C[0,1]. \tag{1}$$

Prove that then there exists a positive Borel measure μ on $[0,1]$, with $\mu([0,1]) = 1$, so that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} f(x_j) = \int_{[0,1]} f(x)\, d\mu(x)$$

for all $f \in C[0,1]$.

(b) Show that if $x_j \to x \in [0,1]$, then (1) holds. What measure μ is obtained in this case?

Please give complete arguments and use good mathematical notation.