MA\QUALIFIER EXAM, TOPOLOGY, SU 2008 (52 PTS)

NAME ____

I. BASICS.

1. Let X be a space and A, $B \subset X$. Show that $(A \cup B)' \subset A' \cup B'$. (3 pts)

2. Suppose that $f : X \to Y$ is a function between spaces X and Y and f is continuous at x for all $x \in X$. Prove that for each open subset V of Y, $f^{-1}(V)$ is open in X. (3 pts)

3. Let $f : X \to Y$ and $g : Y \to Z$ be maps. Prove that $g \circ f : X \to Z$ is a map. (3 pts)

4. Let A be a closed subset of a space X, suppose that (x_n) is a sequence in A, and (x_n) converges to an element x of X. Show that $x \in A$. (3 pts)

5. Let X be a connected space and $f : X \to Y$ a surjective map. Prove that Y is connected. (3 pts)

6. Suppose that A, B are compact subspaces of a space X. Show that $A \cup B$ is compact. (3 pts)

7. Let X, Y be nonempty spaces where Y is compact. Prove that the coordinate projection $\pi_X : X \times Y \to X$ is a closed map. (You may use the Tube Lemma). (3 pts)

8. Suppose that X is a space, $f : X \to Y$ is a surjective function, and Y is given the quotient topology induced by the function f. Prove that f is a map. Let Z be a space and $g : Y \to Z$ a function. Prove that g is a map iff $g \circ f$ is a map. (3 pts)

II. MORE ADVANCED.

1. Let $f : X \to Y$ be a map between spaces X and Y and suppose that A is an F_σ-subset of Y. Show that $f^{-1}(A)$ is a F_σ-subset of X. (3 pts)

2. Let X be a space and $\mathcal{F}$ a locally finite collection of closed subsets of X. Prove directly that $\bigcup \mathcal{F}$ is closed in X. (3 pts)

3. Let X be a regular space and $\mathcal{B}$ a base for its topology. Suppose that for each pair U, $V \in \mathcal{B}$ with $\overline{U} \subset V$, there is a map $f_{U,V} : X \to [0,1]$ with $f_{U,V}(\overline{U}) \subset \{0\}$ and $f_{U,V}(X \setminus V) \subset \{1\}$. Prove that $\{f_{U,V} \,|\, U, V \in \mathcal{B}, \overline{U} \subset V\}$ separates points and closed sets. (We will use this definition. A collection $\mathcal{F}$ of maps of X to $[0,1]$ *separates points and closed sets* if for each $x \in X$ and closed subset A of X with $x \notin A$, there exists $f \in \mathcal{F}$ with $f(x) = 0$ and $f(A) \subset \{1\}$.) (3 pts)

4. Let X be a locally compact space, Y a space, and $f : X \to Y$ a function. Prove that f is a map if $f|K : K \to Y$ is a map for each compact subset K of X. (3 pts)

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5. Let X be a space and $\{Y_\alpha \mid \alpha \in \Gamma\}$ an indexed collection of spaces. Suppose that $f : X \to \prod\{Y_\alpha \mid \alpha \in \Gamma\}$ is a function. Prove that f is a map iff $\pi_\alpha \circ f : X \to Y_\alpha$ is a map for all $\alpha \in \Gamma$. (3 pts)

6. Let (X, d) be a complete metric space. Suppose that (D_n) is a sequence of closed subsets of X such that for each $n \in \mathbb{N}$,

(1) $D_{n+1} \subset D_n$, and
(2) there exists $x_n \in D_n$ such that $D_n \subset B_d(x_n, \frac{1}{2n})$.

Prove that (x_n) is a Cauchy sequence, (x_n) has a limit $x \in X$, and x is an element of D_n for all $n \in \mathbb{N}$. (3 pts)

7. Suppose that B is a space, S is a subspace of B, and $a \in S$. Assume that $\pi_1(B, a)$ is a finite group and $\pi_1(S, a)$ is an infinite group. Prove that there does not exist a retraction r of B onto S. (3 pts)

8. Let X be a set and d the discrete metric on X. Prove that (X, d) is a complete metric space. (3 pts)

9. Let X be a Hausdorff space, A a compact subspace of X, and $p \in X \setminus A$. Prove that there exist nbhds U of p and V of A such that $U \cap V = \emptyset$. (3 pts)

End of Exam. Please turn in this sheet with your solutions all stapled together.