Algebra Qualifying Exam, Spring 2009

1. State the Fundamental Theorem of Algebra.

2. Find the eigenvalues and the eigenvectors of the following matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

3. Let V be a finite dimensional vector space over a field F and let $T : V \to V$ be a linear map such that $Ker(T) \cap Im(T) = \{0\}$. Show that

$$V = Ker(T) \oplus Im(T).$$

4. State Spectral Theorem for the finite dimensional vector spaces, with an inner product, over $\mathbb{R}$ or $\mathbb{C}$.

5. Let V be an n-dimensional vector space over $\mathbb{R}$, with an inner product, and let $A \subseteq End(V)$ be a vector subspace of mutually commuting self adjoint linear maps $L : V \to V$. What is the maximal dimension of A?

6. State the Structure Theorem for the finitely generated abelian groups.

7. How many abelian groups, up to an isomorphism, of order 27 are there?

8. Let G be a group in which all elements other than the identity have order 2. Prove that G is abelian.

10. State Sylow Theorem.

11. How many elements of order 7 are there in a simple group of order 168?

12. Decompose $\mathbb{R}^2$ into the disjoint union of orbits under the action of the orthogonal group O_2.

13. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a ring homomorphism such that $\phi(1) = 1$. Show that ϕ is the identity map. (Here $\mathbb{R}$ is the field of real numbers.)

14. Suppose R is an integral domain and $F \subseteq R$ is a subring that is a field. Prove that if the dimension of R, viewed as a vector space over F, is finite that R is also a field.

15. What is a necessary and sufficient condition on a positive integer N so that the positive square root $\sqrt{N} \in \mathbb{Q}(2^{1/3})$?

16. Give an example of two different algebraically closed fields E and F, such that E is a subfield of F.

17. Give an example of a unique factorization domain which is not a principal ideal domain.

18. Give an example of a ring R and a prime ideal $I \subseteq R$ which is not maximal.