Topology Qualifying Exam **Name:**
May 22, 2009

NOTE: This test does not constitute a syllabus for the topology qualifying exam. In particular, the August 2009 exam may look very dissimilar to this test. To get a better picture of what that exam might be like take the 'average' of all of the previous topology qualifying exams posted at www.math.ou.edu/grad/qualexam.html. This web page also has a topics syllabus for the 2009 exam.

Instructions: Provide justification for each of your answers and make your arguments clear, but try to avoid excessive detail unless they are specifically called for in the problem.

PART I: Work each of the following problems:

1. Let X be an infinite set. Show that the collection of all sets $X - F$ where F is a finite subset of X with an even number of elements is a basis for the finite complement topology on X.

2. Let $f : X \to Y$ be a function between topological spaces. Suppose that C_1 and C_2 are closed sets in X with $X = C_1 \cup C_2$. Show that f is continuous if and only if $f|_{C_1}$ and $f|_{C_2}$ are continuous.

3. (a) Give an example of a sequence of real numbers which converges in the Euclidean topology but not in the lower limit topology.
(b) Show that if a sequence of real numbers converges in the lower limit topology then it converges in the Euclidean topology.

4. Let X be a linearly ordered set with the order topology. An *interval* is a subset $I \subset X$ with the property that if a and b are elements of I and $a < z < b$ then z is an element of I.
(a) Show that a connected subspace of X is an interval.
(b) Show that a path connected subspace of X is an interval.

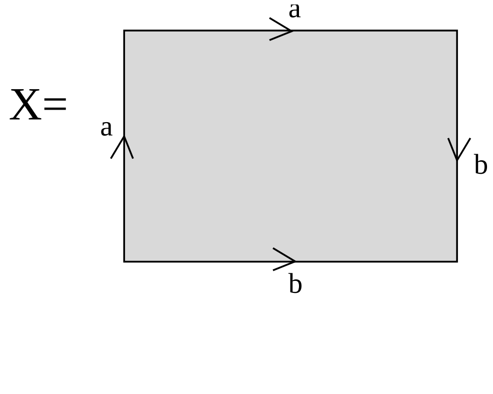
5. Prove that if X_α is Hausdorff for all $\alpha \in J$ then $\prod_{\alpha \in J} X_\alpha$ is Hausdorff (with the product topology). Explain why the converse is not quite true.

6. Let X, Y and Z be topological spaces and let $f_1 : X \to Y$, $f_2 : X \to Y$, $g_1 : Y \to Z$ and $g_2 : Y \to Z$ be continuous functions. Show that if f_1 is homotopic to f_2 and g_1 is homotopic to g_2 then $g_1 \circ f_1$ is homotopic to $g_2 \circ f_2$.

7. Sketch a proof that if x_0 and x_1 are elements in the same path component of a space X then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

8. Describe and sketch the universal covers of $S^2 \vee S^2$ and $\mathbb{R}P^2 \vee S^2$.

9. Let X be a topological space and $Y \subset X$.
(a) Define the terms: Y is a retract of X, and Y is a deformation retract of X.
(b) Show that $S^1 \times \{0, 1\}$ is not a retract of $S^1 \times I$.
(c) Give an example where Y is a retract of X but not a deformation retract of X.
(d) Let X be the 2-dimensional cell complex pictured below. Show that the 1-skeleton X^1 is not a retract of X.



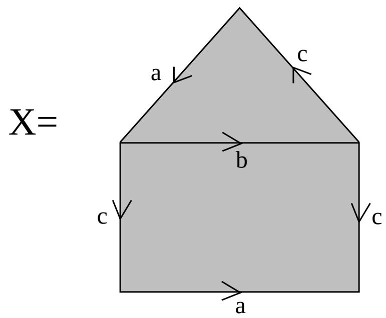
PART II: Do three problems from this section:

10. Show that a compact subspace of a Hausdorff space is closed but that a compact subspace of a non-Hausdorff space need not be closed.

11. State the Unique Path Lifting Lemma from covering space theory and give a brief outline of its proof.

12. Show that a metrizable space is T1 and normal.

13. Explain how to apply van Kampen's Theorem to derive a presentation for the fundamental group of the cell complex X formed from two 2-cells by identifying 1-cells as pictured below. Describe the fundamental group as a well-known group.



PART III: Choose two problems to work:

14. Let X be a T1 space. Show that X is regular if and only if given a point $x \in X$ and a neighborhood U of x there is a neighborhood V of x such that $\overline{V} \subset U$. Is the hypothesis that X be T1 necessary in this statement?

15. A topological space is *σ-compact* if it is a countable union of compact subspaces.
(a) Show that a σ-compact space is Lindelöf.
(b) Is $\mathbb{R}_\ell$ (the real line with the lower limit topology) σ-compact?

16. A *topological pair* (X, Y) consists of a space X and a subspace $Y \subset X$. Two pairs (X_1, Y_1) and (X_2, Y_2) are *homeomorphic* if there is a homeomorphism $f : X_1 \to X_2$ with $f(Y_1) = Y_2$.
(a) If (X_1, Y_1) and (X_2, Y_2) are homeomorphic pairs then show that Y_1 is homeomorphic to Y_2.
(b) Let $X = \mathbb{R}^2 - \{(0,0)\}$ with the Euclidean topology and let C_1 and C_2 be the subspaces $C_1 = \{(x,y) \mid x^2 + y^2 = 1\}$ and $C_2 = \{(x,y) \mid (x-1)^2 + (y-1)^2 = 1\}$. Show that the pairs (X, C_1) and (X, C_2) are not homeomorphic.

17. Let X be a T1 space.
(a) Define what it means for X to be completely regular.
(b) Show that subspaces of completely regular spaces are completely regular.
(c) Show that if X is normal then X is completely regular and that if X is completely regular then X is regular.
(d) Prove that a locally compact Hausdorff space is completely regular.