Algebra Qualifying Exam, Spring 2009

1. Find the greatest common divisor of the polynomials $p(x) = x^4 + x^3 + 2x^2 + x + 1$ and $q(x) = x^5 + 2x^3 + x$ in $\mathbb{R}[x]$.

2. Let $A \in M_{kk}(\mathbb{C})$, $B \in M_{kn}(\mathbb{C})$, and $D \in M_{nn}(\mathbb{C})$. Show that

$$det\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = det(A)det(D).$$

3. Let V be a finite dimensional inner product space over $\mathbb{C}$ and let $L : V \to V$ be a self-adjoint map such that $L^2 = 0$. Find L.

4. Let G be a non-abelian group of order p^n, where p is prime and n is a positive integer. Prove that any subgroup of order p^{n-1} is normal.

5. Let G be a free abelian group of rank n and let $H \subseteq G$ be a free abelian group of rank n. Prove that H has finite index in G.

6. Let G be a finite group of order $p^n r$ where the prime p does not divide r. Let $P \subseteq G$ be a Sylow p-subgroup and let N be the normalizer of P in G. Show that the normalizer of N in G is equal to N.

7. How many elements of order 5 are there in a simple group of order 120?

8. Recall the action of the group $SL(2, \mathbb{Z})$ on $\mathbb{R} \cup \{\infty\}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x) = \begin{cases} \dfrac{ax+b}{cx+d} \text{ if } cx+d \neq 0, \\ \infty \text{ if } cx+d = 0, \\ \dfrac{a}{c} \text{ if } x = \infty. \end{cases}$$

Show that the orbit of 0 is equal to $\mathbb{Q} \cup \{\infty\}$.

9. Give an example of an integral domain which is not a unique factorization domain.

10. Prove that an integral domain with finitely many elements is a field.

11. What is a necessary and sufficient condition on a positive integer N so that the positive square root $\sqrt{N} \in \mathbb{Q}(2^{1/3})$?

12. Give an example of a countable algebraically closed field.

13. Let F be a field of characteristic $p > 0$. Show that the map

$$F \ni x \to x^p \in F$$

is a ring homomorphism.

14. Define a composition series of a group.