Topics in Real Analysis I & II (MATH 5453–5463)

for the PhD Qualifying Examination in Analysis

(Prepared by K. A. Grasse, Spring 2010)

Disclaimer: the list of topics presented below is intended to be reasonably representative, but is not guaranteed to be exhaustive. All of these topics are covered in the three references listed below.

1. **The Real Number System:** basic properties of the real number system $\mathbb{R}$ and the extended real number system $\overline{\mathbb{R}}$, open sets, closed sets, cluster points, structure theorem for open sets, infima and suprema, completeness of $\mathbb{R}$, perfect sets, the Cantor set and Cantor function. Notions of limit inferior and limit superior for sequences of extended real numbers (a_n) (that is, definitions and properties of $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$) and for functions $f : \mathbb{R} \to \mathbb{R}$ ($\liminf_{x\to a} f(x)$ and $\limsup_{x\to a} f(x)$).

2. **Metric Spaces:** elements of the theory of metric spaces, open sets, closed sets, F_σ and G_δ sets, oscillation of a function at a point, continuous functions, uniform continuity, pointwise and uniform convergence of functions defined on metric spaces, the uniform limit of continuous functions is continuous. Cauchy sequences, complete metric spaces, the Baire category theorem (including notions of first and second category sets) and applications.

3. **Abstract Measure Spaces:** algebras and σ-algebras of sets, measures defined on algebras and σ-algebras. Elementary properties of measures. Complete measure spaces and the completion of a (non-complete) measure space. Definition of the σ-algebra $\boldsymbol{\mathcal{B}}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$ and the σ-algebra $\boldsymbol{\mathcal{B}}(X)$ of Borel subsets of any metric space X.

4. **The Riemann Integral:** the definition and basic properties of the Riemann integral of a bounded function $f : [a, b] \to \mathbb{R}$; the criterion for Riemann integrability in terms of upper and lower Riemann sums; Riemann integrability of increasing and continuous functions.

5. **Measurable Functions:** the definition and basic properties of measurable functions, the structure theorem for non-negative measurable functions. Sequences of measurable functions, pointwise limit of a sequence of measurable functions is measurable, modes of convergence (pointwise, pointwise a. e., uniform, almost uniform, convergence in measure). Egoroff's theorem, Lusin's theorem, and the Weierstrass approximation theorem. Riesz's theorem (and its consequences) concerning sequences of measurable functions that are Cauchy in measure.

6. **Extensions of Measures:** the outer measure on the power set $\boldsymbol{\mathcal{P}}(X)$ (X here is any set) induced by a measure on an algebra of subsets of X, basic properties of the outer measure (for example, sub-additivity). Extensions of measures from algebras of sets to σ-algebras of sets via the Carathéodory extension theorem; conditions under which the Carathéodory extension is unique.

7. **Lebesgue-Stieltjes Measures:** definition of the LS-measure of an elementary (that is, right semi-closed) interval corresponding to an increasing, right-continuous function $g : \mathbb{R} \to \mathbb{R}$, characterization of of the σ-algebra of LS-measurable sets $\boldsymbol{\mathcal{M}}_g$in terms of the LS outer measure, and the extension (via the Carathéodory theorem) of the measure from the algebra of finite unions of elementary intervals to the σ-algebra $\boldsymbol{\mathcal{M}}_g$, characterization of sets in $\boldsymbol{\mathcal{M}}_g$ in terms of approximation properties by open sets, closed sets, F_σ sets, and G_δ sets. The special case of (standard) Lebesgue measure (when $g(x) \equiv x$), translation invariance of Lebesgue measure, the existence of non-Lebesgue measurable subsets of $\mathbb{R}$.

8. **The Abstract Lebesgue Integral:** the definition and basic properties. Monotone convergence theorem, Fatou's lemma, dominated convergence theorem. Equality of the Riemann and Lebesgue integrals (with respect to the standard Lebesgue measure on $\mathbb{R}$) for Riemann integrable functions $f : [a, b] \to \mathbb{R}$; the characterization of the Riemann integrability of a bounded function $f : [a, b] \to \mathbb{R}$ in terms of the Lebesgue measure of its set of discontinuities.

9. **Differentiation:** Vitali covering theorem, Dini derivates, monotone functions are differentiable a. e. Functions of bounded variation and their characterization as the difference of increasing functions, absolutely continuous functions, AC $\Rightarrow$ BV, characterization of absolutely continuous functions as the indefinite Lebesgue integrals of their derivatives.

10. L^p**-Spaces:** the definition of the L^p spaces for both $1 \leq p < \infty$ and $p = \infty$, Hölder and Minkowski inequalities, L^p-convergence and its relation to other types of convergence (for example L^p convergence $\Rightarrow$ convergence in measure). Riesz-Fischer theorem (completeness of $L^p(\mu)$). The Vitali convergence theorem (characterization of convergence in L^p when $1 \leq p < \infty$). Dense subsets of $L^p[a, b]$ (simple functions, continuous functions, polynomials).

11. **Product Measures:** Definition of product measures, monotone class lemma, the existence of the product measure and conditions under which it is uniquely determined by its values on measurable rectangles, Tonelli and Fubini theorems. Lebesgue measure on $\mathbb{R}^n$.

12. **Signed Measures:** Hahn and Jordan decomposition theorems. Absolutely continuous and singular pairs of measures, the Radon-Nikodým theorem and the Lebesgue decomposition theorem.

13. **Dual Spaces:** notions of a normed vector space, the dual space of a normed linear space, bounded linear functionals and their norms. The Riesz representation theorem for $L^p(\mu)$, $1 \leq p < \infty$.

References: H. L. Royden, *Real Analysis (3rd Ed.)*
W. Rudin, *Real and Complex Analysis (2nd Ed.)*
R. Bartle, *The Elements of Integration*