Algebra Qualifying Exam, Spring 2011

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All rings are associative with 1 and all modules are unital left modules.

1. Prove or disprove:
   a. If H is a normal subgroup of G and K is a normal subgroup of H then K is normal in G.
   b. If a group G has a subgroup of finite index then G has a normal subgroup of finite index.
   c. A finite p-group has a nontrivial center.

2. Show that there are no simple groups of order 48 or 120.

3. If the positive integer k divides the order of a finite nilpotent group G then prove that G has a normal subgroup of order k.

4. Show that the matrix ring $\mathrm{M_n}(\mathbb{D})$ is simple when $\mathbb{D}$ is a division ring.

5. If $\mathbb{F}$ is a field of characteristic p and G is a finite p-group then show that the radical $J(\mathbb{F}G)$ of the group algebra $\mathbb{F}G$ is a maximal ideal of $\mathbb{F}G$.

6. Prove that if I and J are ideals of the commutative ring R with $I + J = R$ then R/IJ is isomorphic to $R/I \times R/J$.

7. Give examples of finite Galois extensions $\mathbb{K}$ of the field of rational numbers $\mathbb{Q}$ so that $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ is isomorphic to:
   a. The dihedral group of order 8.
   b. An elementary abelian 2-group of arbitrary rank.
   c. Any cyclic group of prime order.

8. If $\mathbb{K}$ is the splitting field in $\mathbb{C}$ of $x^p - 2$ over $\mathbb{Q}$ with p prime then show that $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ is isomorphic to the semidirect product $\mathbb{Z}/p\mathbb{Z} \rtimes (\mathbb{Z}/p\mathbb{Z})^{\times}$.

9. Prove that for positive integers $m < n$: if $q^m - 1$ divides $q^n - 1$ for some prime power q then $k^m - 1$ divides $k^n - 1$ for every integer k.