University of Oklahoma Department of Mathematics
Real Analysis Qualifier Exam
May 16, 2011

Directions: Answer each question on a separate page, writing your ID number (not your name) in the upper right corner of each page and the problem number in the upper left corner of each page. Completely justify your work and state which theorems or results you are citing. You have 3 hours to complete this exam.

Lebesgue measure is denoted by m, the real line is denoted by $\mathbb{R}$.

1. Let $f, g : \mathbb{R} \to \mathbb{R}$.

   (a) Show that if f is Borel measurable and g is Lebesgue measurable, then $f \circ g$ is Lebesgue measurable.

   (b) Give an example of a Lebesgue measurable function f and a continuous function g such that $f \circ g$ is not Lebesgue measurable.

2. Let f be an integrable function on the measure space $(X, \mathcal{M}, \mu)$ and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of disjoint measurable subsets of X. Let $A = \displaystyle\bigcup_{n=1}^{\infty} A_n$. Prove that

$$\int_A f \,\mathrm{d}\mu = \sum_{n=1}^{\infty} \int_{A_n} f \,\mathrm{d}\mu.$$

3. Given $E \subseteq \mathbb{R}$, prove that the following statements are equivalent:

   (a) E is Lebesgue measurable.

   (b) For all $\epsilon > 0$ there is an open set $\mathcal{O}$ containing E such that $m^*(\mathcal{O} \sim \mathcal{E}) < \epsilon$.

   (c) There is a G_δ set G containing E such that $m^*(G \sim E) = 0$.

4. For $K > 0$, let M_K be the space of all Lipschitz functions on $[0, 1]$ with constant K, i.e. all real-valued functions f on $[0, 1]$ such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in [0, 1]$.

   (a) Show that M_K is closed under the usual supremum metric on $C[0, 1]$.

   (b) Show that D_K, the set of all differentiable functions g on $[0, 1]$ such that $|g'| \leq K$, is contained in M_K for any $K > 0$.

   (c) Show that $M = \bigcup_{K>0} M_K$ is *not* closed.

   (d) Show that $\overline{M}$, the closure of M, equals $C[0, 1]$.

5. Let

$$f(x,y) = \begin{cases} y^{-2} & \text{if } 0 < x < y < 1 \\ -x^{-2} & \text{if } 0 < y < x < 1 \\ 0 & \text{otherwise on } [0,1] \times [0,1] \end{cases}.$$

(a) Compute

$$\int_0^1 \int_0^1 f(x,y)\,\mathrm{d}m(y)\,\mathrm{d}m(x) \qquad \text{and} \qquad \int_0^1 \int_0^1 f(x,y)\,\mathrm{d}m(x)\,\mathrm{d}m(y).$$

(b) State Fubini's Theorem. Explain why it does or does not apply to this example.

6. (a) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then $\{x_n\}_{n=1}^{\infty}$ converges to the real number L if and only if every subsequence of $\{x_n\}_{n=1}^{\infty}$ has in turn a subsequence that converges to L.

(b) Assume E has finite Lebesgue measure. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue measurable functions on E and f a Lebesgue measurable function on E such that f and each f_n is finite a.e. on E. Prove that $\{f_n\}_{n=1}^{\infty} \to f$ in measure on E if and only if every subsequence of $\{f_n\}_{n=1}^{\infty}$ has in turn a further subsequence that converges to f pointwise a.e. on E.

7. Let $\{f_n\}_{n=1}^{\infty}$ and f be in $L^2(\mathbb{R})$ and let $\|\cdot\|_2$ be the L^2 norm.

(a) Prove that if for all functions $g \in L^2(\mathbb{R})$ we have

$$\lim_{n\to\infty} \int f_n g\,\mathrm{d}x = \int f g\,\mathrm{d}x$$

and if $\|f_n\|_2 \to \|f\|_2$, then $\|f_n - f\|_2 \to 0$ as $n \to \infty$.

(b) Give an example to show that part (a) may fail if we do not assume $\|f_n\|_2 \to \|f\|_2$ as $n \to \infty$.