Attempt all six questions. There is an average of 30 minutes per question, so time your answers accordingly.

Q1. (a) Define what it means for a topological space to be *compact* .

(b) Define what it means for a topological space to be *Hausdorff*.

(c) Prove that the continuous image of a compact space is compact.

(d) What can you conclude about a continuous bijection from a compact space to a Hausdorff space? (no proof necessary)

(e) Give the definition of the *quotient topology*.

(f) Give the unit disk $D^2 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\}$ and its boundary circle $S^1 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$ the subspace topology from $\mathbb{R}^2$. Give a detailed proof that the quotient space D^2/S^1 (obtained by identifying all points of S^1 to a single point) is homeomorphic to the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$ with the subspace topology inherited from $\mathbb{R}^3$. Identify (by name or brief statements) the results that you use in your proof.

Q2. Let X be a topological space and $A \subset X$ be a subspace.

(a) Define what it means for A to be a retract of X.

(b) Define what it means for A to be a deformation retract of X.

(c) Is $\{0, 1\}$ with the discrete topology a retract of the unit interval $[0, 1]$ with the usual topology (as a subspace of $\mathbb{R}$)? Give a reason for your answer.

(d) Is the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2+y^2 = 1\}$ with the usual topology (as a subspace of $\mathbb{R}^2$) a retract of the unit disk $D^2 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2+y^2 \leq 1\}$ with the usual topology (as a subspace of $\mathbb{R}^2$)? Give a reason for your answer.

(e) Let $S^1 \vee S^1$denote the CW complex consisting of one 0-cell and two 1-cells, and let S^1 denote the subcomplex consisting of one 0-cell and one 1-cell. Is S^1 a retract of $S^1 \vee S^1$? Give a reason for your answer.

(f) Let $S^1 \subset S^1 \vee S^1$ be as described in the previous part. Is S^1 a deformation retract of $S^1 \vee S^1$? Give a reason for your answer.

Q3. Let $\{X_\alpha \,|\, \alpha \in J\}$ be an indexed collection of topological spaces X_α.

(a) Define the product $\prod_{\alpha \in J} X_\alpha$ as a set.

(b) Define the product topology on $\prod_{\alpha \in J} X_\alpha$.

(c) Prove that the projection maps $P_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$ are continuous.

(d) Prove that a function $f : Z \to \prod_{\alpha \in J} X_\alpha$ is continuous iff $P_\alpha \circ f$ are continuous.

(e) If each of the X_α are connected, is $\prod_{\alpha \in J} X_\alpha$ necessarily connected?

Q4. Let S_Ω denote a well-ordered set whose order type is the first uncountable ordinal Ω. Give $S_\Omega \cup \{\Omega\}$ the ordering in which every element of S_Ω is less than Ω, and consider the corresponding order topology.

(a) Describe basis elements of the order topology which contain the point Ω.

(b) Prove that Ω is a limit point of S_Ω.

(c) Prove that every countable subset $A \subset S_\Omega$ has an upper bound in S_Ω.

(d) Prove that no sequence in S_Ω converges to Ω.

(e) Is $S_\Omega \cup \{\Omega\}$ with the order topology metrizable? Give a reason for your answer.

Q5. State the fundamental theorem (Galois correspondence) about the existence of covering spaces of a space X and their connection with subgroups of $\pi_1(X)$.

(a) Recall that the free group $F_{\{a,b\}}$ is the fundamental group of a wedge of two circles $S^1_a \vee S^1_b$. Use the fundamental theorem to prove that every subgroup of the free group $F_{\{a,b\}}$ is free.

(b) Draw covering spaces of $S^1_a \vee S^1_b$ corresponding to the following subgroups of $F_{\{a,b\}}$.

i. $\langle a \rangle$

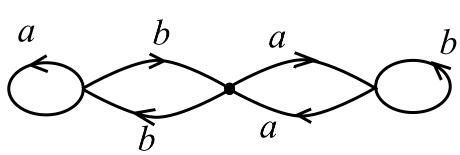
ii. $\langle a, b^2, bab \rangle$

iii. $ker(f)$ where $f : F_{\{a,b\}} \to \langle t | \ \rangle$ is defined by $f(a) = t$ and $f(b) = 1$. Here $\langle t | \ \rangle$ denotes the infinite cyclic group generated by t and 1 denotes the identity element.

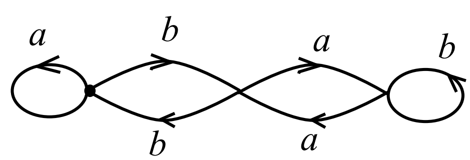
iv. $ker(h)$ where $h : F_{\{a,b\}} \to S_3$ is the homomorphism to the symmetric group $S_3 = \text{Perm}(\{1, 2, 3\})$ defined by $h(a) = (12)$ and $h(b) = (23)$. Note that (12) and (23) are cycle notation descriptions of permutations of the set $\{1, 2, 3\}$.

(c) Write down subgroups of $F_{\{a,b\}}$ corresponding to the following two based covering spaces of $S^1_a \vee s^1_b$. Note that the base point is indicated by a heavy dot in each case.

i)...



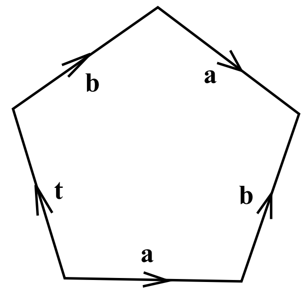
ii)...



(d) Prove that a nontrivial, normal subgroup of $F_{\{a,b\}}$ which is of infinite index is not finitely generated. Say where all the hypotheses in the statement fit into your proof.

(e) Give examples of finitely generated subgroups of $F_{\{a,b\}}$ which satisfy two of the three hypotheses above (you should give three examples in all).

Q6. (a) Let X be the cell complex obtained by attaching the 2–cell in the figure below to the wedge of three circles, $S^1_a \vee S^1_b \vee S^1_t$. Use van Kampen's theorem to compute $\pi_1(X)$.



(b) Say whether the group $\pi_1(X)$ is finite or infinite. Give a reason for your answer.

(c) Say whether the group $\pi_1(X)$ is abelian or not. Give a reason for your answer.

(d) Is there a retraction from X to $S^1_a \vee S^1_b$? Construct an explicit retraction (quoting whatever theorems from general topology that may be necessary to perform the construction), or prove that none exists.

(e) Is there a retraction from X to S^1_t? Construct an explicit retraction (quoting whatever theorems from general topology that may be necessary to perform the construction), or prove that none exists.