Algebra Qualifying Exam   Name:
August 11th, 2014

Instructions: Provide justification for each of your answers and make your arguments clear, but try to avoid excessive detail. Complete any **7** out of **8** for full credit.

1. (a) State and prove the Orbit–Stabilizer theorem for a finite group G acting on a set X.

(b) Let G be a group of order 27 acting on a set X of size $|X| = 50$. Show that there are at least two elements of X which are fixed by G i.e., there are elements x_i, x_j distinct each with isotropy group G.

2. Indicate whether the following statements are TRUE or FALSE. If you believe a given statement is True, then provide a short proof; if False, then construct a counterexample.

(a) If the ring R is a PID, then in R every prime ideal is a maximal ideal.

(b) If a group G has the property that every proper subgroup is abelian, then G must also be abelian.

(c) If a group G has even order, then it must contain a subgroup of index 2.

3. Let R be the commutative ring $R = \mathbf{Z}[x]$. Consider the ideal $\mathcal{I} = (2, x^2 + x + 1) \subseteq R$.

(a) Is $\mathcal{I}$ a maximal ideal? Identify (with proof/explanation) the quotient ring $R/\mathcal{I}$.

(b) Let P be a prime ideal of R such that $P \cap \mathbf{Z} = \{0\}$. Show that P is a principal ideal.

4. Let R be a commutative ring with 1 and let $\mathcal{I}$ be an ideal of R. Recall that $\mathcal{I}$ is called a *radical ideal* if its radical $\mathcal{R}(\mathcal{I}) = \{r \in R : r^n \in \mathcal{I} \text{ for some } n\} = \mathcal{I}$ equals itself. Show that every prime ideal of R is a radical ideal.

5. Prove that there is no simple group of order $108 = 4 \cdot 27$. (*Hint*: Let P be a Sylow 3-subgroup of such a group G. Consider the left action of G on the set of cosets G/P which is a transitive action. Conclude that one has a non–trivial homomorphism, $G \to S_4$, and thereby conclude G has a proper, normal subgroup.)

6. Let $p(x) \in \mathbf{Q}[x]$ be an irreducible polynomial of degree 4 and suppose K is its splitting field over $\mathbf{Q}$. Suppose $p(x)$ has exactly two real roots, then show that the Galois group, $\mathrm{Gal}(E/\mathbf{Q})$ cannot be A_4, the alternating group.

7. Let E be the splitting field of the polynomial $f(x) = x^p - x - a \in \mathbf{F}_p[x]$, where p is prime and $a \neq 0$ is an element of $\mathbf{F}_p$. Show that $f(x)$ is irreducible by showing that if $\alpha \in E$ is a root of $f(x)$, then so is $\alpha + 1$. Use this fact to also show that $\mathrm{Gal}(E/\mathbf{F}_p) \cong \mathbf{Z}/p$.

8. Let $\zeta = e^{2\pi i/7}$ denote a primitive 7-th root of unity and let $K = \mathbf{Q}(\zeta)$ be the associated cyclotomic field extension with Galois group $\mathrm{Gal}(K/\mathbf{Q})$. Let $\alpha = \zeta + \zeta^2 + \zeta^4 \in K$.

(a) Describe explicitly a generator of the Galois group $\mathrm{Gal}(K/\mathbf{Q})$.

(b) Show that $[\mathbf{Q}(\alpha) : \mathbf{Q}] = 2$.