Qualifying Exam: Analysis

Name:...

1. **(3+2 points)** (a) Let μ be a σ-finite measure on $(X, \mathcal{M})$ with $\mu(X) = \infty$. Show that for every $C > 0$, there exists an $E \in \mathcal{M}$ with $C < \mu(E) < \infty$.
   (b) Give an example that shows that this need not hold if we drop the assumption that μ is σ-finite.

2. **(2+2 points)** Let $f \in L^1(X, d\mu)$.
   (a) Show that $\{x \in X : |f(x)| > 0\}$ is a σ-finite set (that is, it can be written as a countable union of sets of finite measure).
   (b) Show that it is possible to have
   $$\mu\left(\{x \in X : |f(x)| > 0\}\right) = \infty$$
   (please give a concrete example of a function $f \in L^1$ on a space X where this happens).

3. **(2+2+2+2 points)** Consider the functions $f_n = n^2 \chi_{(0,1/n)}$ (note that $f_n \in L^1(\mathbb{R})$). Does the sequence f_n converge
   (a) pointwise almost everywhere?
   (b) in L^1?
   (c) in measure?
   (d) in $\mathcal{D}'(\mathbb{R})$?

   In those cases where it does converge, please also identify the limit.

4. **(5 points)** Consider the increasing, right-continuous function
   $$F(x) = \begin{cases} 0 & x < 0 \\ 1 + x & x \geq 0 \end{cases},$$
   and let $\nu = \nu_F$ be the associated Borel measure on $\mathbb{R}$ (so $\nu((a, b]) = F(b) - F(a)$). Find the Lebesgue decomposition of ν with respect to:
   (a) $\mu = m$;
   (b) $\mu = \delta$, the Dirac measure at 0;
   (c) the Cantor measure μ.

   In other words, write ν as $\nu = \rho + \lambda$ with $\rho \ll \mu$, $\lambda \perp \mu$. Please clearly identify the measures λ, ρ in each case.

5. **(3 points)** Let $E \subset \mathbb{R}^n$ be a Borel set with $m(E) > 0$. Show that for every $\epsilon > 0$, there exists an open ball $B = B(r, x)$ so that
   $$m(E \cap B) \geq (1 - \epsilon) m(B).$$

6. **(5 points)** Let $f \in L^1(\mathbb{R})$. Prove that

$$\int_{-1}^{1} \widehat{f}(\xi) e^{2\pi i \xi x}\, d\xi = (f * D)(x),$$

where

$$D(t) = \frac{\sin 2\pi t}{\pi t}.$$

Please don't just give the formal calculation; carefully justify all steps.

Suggestion: Use the definition of the Fourier transform and then Fubini-Tonelli to evaluate the resulting iterated integral.

7. **(3+3 points)** Please give an example of a function $f : (0, \infty) \to \mathbb{C}$ with the following properties:
(a) $f \in L^p(0, \infty)$ for $2 \le p \le \infty$, but $f \notin L^p(0, \infty)$ if $1 \le p < 2$;
(b) $f \in L^p(0, \infty)$ for $2 < p < 4$, but not for p outside this range

8. **(3+3 points)** Let μ_n be a sequence of finite Borel measures on $[0, 1]$.
(a) Suppose that

$$\lim_{n\to\infty} \int_{[0,1]} f(x)\, d\mu_n(x)$$

exists for every $f \in C[0, 1]$. Show that then there exists another finite Borel measure μ on $[0, 1]$ so that

$$\lim_{n\to\infty} \int_{[0,1]} f(x)\, d\mu_n(x) = \int_{[0,1]} f(x)\, d\mu(x)$$

for all $f \in C[0, 1]$.
(b) Show that

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{j/n}$$

satisfies the assumptions from part (a) (δ_x denotes the Dirac measure at x), and identify the limit measure μ.

9. **(5 points)** Let $F \in \mathcal{D}'(\mathbb{R})$ be the distribution generated by the L^1_{loc} function $F(x) = \ln |x|$. Prove that (in $\mathcal{D}'(\mathbb{R})$)

$$F' = \mathrm{PV} - \frac{1}{x}.$$

(Recall that this latter distribution was defined as

$$\left\langle \mathrm{PV} - \frac{1}{x}, \phi \right\rangle = \lim_{y\to 0+} \int_{|x|>y} \frac{\phi(x)}{x}\, dx.)$$

Suggestion: As the first step, show that if f is a bounded measurable function, then

$$\lim_{y\to 0+} \int_{-y}^{y} f(x) \ln |x|\, dx = 0.$$

Then use this fact and integration by parts to compute $\langle F', \phi \rangle$.

Please give complete arguments and use good mathematical notation.