Qualifying Exam: Analysis

Name:...

1. **(5 points)** Let μ, ν be finite Borel measures on $\mathbb{R}$ and assume that

$$\mu((-\infty, a)) = \nu((-\infty, a)) \qquad \text{for all } a \in \mathbb{R}.$$

   Show that then $\mu(B) = \nu(B)$ for all Borel sets $B \subset \mathbb{R}$.

   Suggestion: Use the regularity of these measures and the fact that open subsets of $\mathbb{R}$ are countable disjoint unions of open intervals.

2. **(3+4 points)** Let μ be a Borel measure on $\mathbb{R}$, and let $f_n \in L^1(\mathbb{R}, \mu)$ be a sequence of functions with $f_n(x) = 0$ for $|x| \leq n$. In addition, assume that: (i) μ is finite; (ii) $|f_n(x)| \leq 1$.

   (a) Show that $\int_{\mathbb{R}} f_n(x)\, d\mu(x) \to 0$.

   (b) Show that this need not hold if either assumption [(i) or (ii)] is dropped. (Give counterexamples.)

3. **(4+3+1 points)** (a) Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing, absolutely continuous function. Show that if $E \in \mathcal{B}_{\mathbb{R}}$ with $m(E) = 0$, then also $m(F(E)) = 0$, where $F(E) := \{F(x) : x \in E\}$.
   Suggestion: Use outer regularity to approximate E by a union of open intervals. What is $F(I)$ for an interval $I = (a, b)$?

   (b) Now let F be the Cantor function. Show that there exists an $E \subset \mathbb{R}$ with $m(E) = 0$, $m(F(E)) > 0$.
   Hint: Use the description of F from the proof of Proposition 1.22. See especially the information provided in the last three lines of that proof.

   (c) Why do (a) and (b) not contradict each other?

4. **(5 points)** Let F, G be continuous, increasing functions on $\mathbb{R}$, and write μ_F, μ_G for the associated measures (so $\mu_F((-\infty, x]) = F(x)$ etc.). Prove the following *integration by parts* formula:

$$\int_{(a,b)} F(x)\, d\mu_G(x) = -\int_{(a,b)} G(x)\, d\mu_F(x) + F(b)G(b) - F(a)G(a)$$

   Suggestion: Let $T = \{(x, y) : a < x < y < b\} \subset \mathbb{R}^2$ and evaluate

$$\int d\mu_F(x) \int d\mu_G(y)\, \chi_T(x, y)$$

   in two ways. (Please don't forget to justify these manipulations.)

5. **(3+3+3 points)** Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

(a) Show that x will be in the Lebesgue set L_f of f if f is continuous at x.

(b) Show that if $x \in L_f$, then $|f(x)| \le (Hf)(x)$.

(c) Give an example of a function $f \in L^1_{\text{loc}}$ that is not continuous at some point $x \in \mathbb{R}^n$ (even after changing f on a null set), but $x \in L_f$.

Alternatively, you can also deduce the existence of such examples from general facts about L_f and locally integrable functions (rather than construct it explicitly), if you prefer.

6. **(3+3 points)** Let ν be the Borel measure on $\mathbb{R}$ that is generated by the increasing, right-continuous function

$$F(x) = \begin{cases} 0 & x < 0 \\ 2x & 0 \le x \le 1 \\ 5 & x > 1 \end{cases}.$$

(a) Find the Lebesgue decomposition of ν with respect to Lebesgue measure $\mu = m$, that is, find λ, ρ so that $\nu = \lambda + \rho$ and $\lambda \ll \mu$, $\rho \perp \mu$.

(b) Find the Lebesgue decomposition of ν with respect to $\mu = \delta_1$ (so $\mu(\{1\}) = 1$, $\mu(\mathbb{R} \setminus \{1\}) = 0$).

7. **(2+2+2+3 points)** Find all p, $1 \le p \le \infty$, such that $f \in L^p(\mathbb{R})$, for the following functions:

$$\text{(a) } f(x) = 1; \quad \text{(b) } f(x) = \frac{1}{x^2+1}; \quad \text{(c) } f(x) = x^2 e^{-x^2};$$

$$\text{(d) } f(x) = \sum_{n=1}^{\infty} n^{-1/2} (x-n)^{-1/n} \chi_{(n,n+1)}(x)$$

8. **(4 points)** Let $F : \mathbb{R} \to \mathbb{C}$ be absolutely continuous with $F' \in L^p(\mathbb{R})$, $1 \le p < \infty$. Show that there exists a constant $C > 0$ so that

$$|F(x) - F(y)| \le C|x-y|^\alpha \qquad (x, y \in \mathbb{R}),$$

with $\alpha = 1 - 1/p$.

9. **(2+2+4 points)** Recall that in $\mathcal{D}'$, we have that

$$\lim_{\epsilon \to 0+} \frac{1}{x - i\epsilon} = \text{PV} - \frac{1}{x} + i\pi\delta,$$

(a) Deduce from this that

$$\lim_{\epsilon \to 0+} \frac{\epsilon}{x^2+\epsilon^2} = \pi\delta. \tag{1}$$

(b) By formally taking derivatives on both sides, we obtain that

$$\lim_{\epsilon \to 0+} \frac{-2\epsilon x}{(x^2+\epsilon^2)^2} \stackrel{?}{=} \pi\delta'. \tag{2}$$

In general, is it correct to differentiate limiting relations in $\mathcal{D}'$ in this way?

(c) Prove (2) directly. You can make use of (1), if you want.

Please give complete arguments and use good mathematical notation.