Provide justification for all of your answers.

1. (10 points) Let G be a finite group, H a subgroup of G and N a normal subgroup of G. Show that if the order of H is relatively prime to the index of N in G, then $H \subseteq N$.

2. (10 points) Let G be a group of order $231 = 3 \cdot 7 \cdot 11$. Prove that a Sylow 11-subgroup is contained in the center of G.

3. (10 points) Let m, n be positive integers, such that m divides n.

   (a) Show that the natural map $\phi : \mathbb{Z}_n \to \mathbb{Z}_m$ given by $\phi(a + n\mathbb{Z}) = a + m\mathbb{Z}$ is a surjective ring homomorphism.

   (b) If U_n, U_m are the units of $\mathbb{Z}_n$ and $\mathbb{Z}_m$, respectively, show that $\phi : U_n \to U_m$ is a surjective group homomorphism.

4. (10 points) Find all the values of a in $\mathbb{Z}_3$ such that the quotient ring

$$\mathbb{Z}_3[x]/(x^3 + x^2 + ax + 1)$$

   is a field. Justify your answer.

5. (10 points) Show that $\mathbb{Z}(\sqrt{5}) = \{m + n\sqrt{5} | m, n \in \mathbb{Z}\}$ is not a unique factorization domain.

6. (10 points) Let $\alpha = \sqrt{2} + \sqrt{3}$ and $E = \mathbb{Q}(\alpha)$.

   (a) Find the minimal polynomial $m(x)$ of α over $\mathbb{Q}$ and the degree $[E : \mathbb{Q}]$.

   (b) Find the splitting field K of $m(x)$ over $\mathbb{Q}$ and the order of the Galois group $Gal(K/\mathbb{Q})$.

7. (10 points) Let $\epsilon = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$ be a primitive n-th root of 1.

   (a) If $\Phi_n(x)$ is the minimal polynomial of ϵ over $\mathbb{Q}$, show that $\mathbb{Q}(\epsilon)$ is a splitting field of $\Phi_n(x)$ over $\mathbb{Q}$.

   (b) Prove that the Galois group of $\mathbb{Q}(\epsilon)$ over $\mathbb{Q}$ is isomorphic to the group of units of $\mathbb{Z}_n$.