# ANALYSIS QUALIFYING EXAM – SPRING 2016

NAME: . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

**Complete 5 of the problems below. If you attempt more than 5 questions, then clearly indicate which 5 should be graded.**

(Each problem will count for 10 points.)

For full credit you must provide complete arguments, and state domains ranges etc. whenever you introduce functions, variables, sets and so on. You are allowed to (and you should) refer to results we discussed in class – do not reprove basic textbook material – but clearly indicate/cite the results you use.

**A note on the notation used below:** The symbol $\mathscr{L}^p$ refers to the set of all real-valued functions $f$ such that $|f|^p$ is integrable (the value $p = \infty$ is a bit different). Clearly, the setup implicitly assumes a measure space $(X, \mathscr{F}, \mu)$, so that the domain of $f$ is $X$, measurable means $\mathscr{F}$–measurable, and integrable means integrable with respect to $\mu$. Sometimes we write $\mathscr{L}^p(0, \infty)$ or alike to explicitly indicate parts of the underlying measure space – in this example $X = (0, \infty)$. As is common practice, integration with respect to the Lebesgue measure, denoted by Leb, is usually written simply as $\int f(x)\,dx$ instead of $\int f(x)\,\mathrm{Leb}(dx)$.

| Question | Points |
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| 1 | |
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*Date*: January 16, 2016.

1. Given $f\colon \mathbb{R} \to \mathbb{R}$ and $y \in \mathbb{R}$, define $f_y(x) = f(x-y)$. Consider the Lebesgue measure on the Borel sets of $\mathbb{R}$.
   (a) Show that if $f$ is continuous with compact support, then $\lim_{y\to 0} \| f_y - f \|_\infty = 0$.
   (b) Show that if $f \in \mathscr{L}^p(\mathbb{R})$ for some $1 \le p < \infty$, then $\lim_{y\to 0} \| f_y - f \|_p = 0$.
   (c) Prove or disprove by a counterexample: if $f \in \mathscr{L}^\infty$, then $\lim_{y\to 0} \| f_y - f \|_\infty = 0$.

2. Let $(X, \mathscr{F}, \mu)$ be a measure space, and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets in $\mathscr{F}$ such that $\lim_{n\to\infty} \mu(A_n) = 0$.
   (a) Prove that there exists a subsequence $(A_{n_k})_{k\in\mathbb{N}}$ such that $\cap_{m=1}^\infty \cup_{k=m}^\infty A_{n_k}$ has $\mu$–measure zero.
   (b) Consider the special case where $X = \mathbb{R}$, $\mathscr{F}$ are the Borel sets on $\mathbb{R}$, and $\mu$ is the Lebesgue measure. Prove or disprove by a counterexample that $\cap_{m=1}^\infty \cup_{n=m}^\infty A_n$ has Lebesgue measure zero.

3. Consider the Lebesgue measure on the Borel sets of $[0,1]$. Prove by an example or disprove: There exists a function $f$ such that
   - $f \in \mathscr{L}^p$ for all $1 \le p < \infty$
   - $\liminf_{z\to x} f(z) = -\infty$, $\limsup_{z\to x} f(z) = +\infty$, for every $x \in [0,1]$.

4. Let $\mu$ be a finite measure on the Borel sets of $X = [0,1]$ such that $\mu(\{x\}) = 0$ for all $x \in X$. Prove that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\mu(A) \le \epsilon$ for all intervals $A \subset X$ with $\operatorname{diam}(A) \le \delta$.
   Remark: The statement remains true if $X$ is a compact metric space, and $A$ is any Borel set (instead of a metric ball/interval).

5. Let $(X, \mathscr{F}, \mu)$ be a finite measure space, and suppose that $\nu$ is a (finite) signed measure on $(X, \mathscr{F})$. Prove that the following two statements are equivalent
   (a) $\nu(A) = 0$ for all $A \in \mathscr{F}$ for which $\mu(A) = 0$
   (b) for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|\nu(A)| < \epsilon$ for all $A \in \mathscr{F}$ for which $\mu(A) < \delta$.

6. Let $(X, \mathscr{F}, \mu)$ be a measure space. Prove that the following are equivalent:
   (a) There exists $0 < p_1 < p_2 < \infty$ such that $\mathscr{L}^{p_2} \subset \mathscr{L}^{p_1}$.
   (b) $\sup\{\mu(A) \colon A \in \mathscr{F},\ \mu(A) < \infty\} < \infty$, i.e. there exists an $M < \infty$ such that $\mu(A) \le M$ for all $A \in \mathscr{F}$ with $\mu(A) < \infty$.
   (c) For every $0 < p_1 < p_2 < \infty$ it follows that $\mathscr{L}^{p_2} \subset \mathscr{L}^{p_1}$.
   Hint: Since $(c) \implies (a)$ is obvious, one strategy is to show $(c) \implies (a) \implies (b) \implies (c)$.

7. Consider the Lebesgue measure on the Borel set of $\mathbb{R}^n$. Let $A$ be a Borel subset of $\mathbb{R}^n$, and let $f\colon A \to \mathbb{R}$ be an integrable function such that $\int_A f(x)\,dx = r > 0$. Prove that for every $0 < c < r$ there exists a Borel subset $S \subset A$ such that $\int_S f(x)\,dx = c$.

8. Let $(X, \mathscr{F}, \mu)$ be a measure space (not necessarily $\sigma$–finite!). Prove that for any $f\colon X \to [0,\infty]$ measurable (not necessarily integrable!)
$$\int f(x)\,\mu(dx) = \int_0^\infty \mu\{f > t\}\,dt\ .$$
   (Remark: The integral on the right-hand-side is actually an improper Riemann integral, but you may consider it as a Lebesgue integral.)

9. Consider a $\sigma$–finite measure space $(X, \mathscr{F}, \mu)$, and fix a measurable function $g\colon X \times X \to \mathbb{R}$ such that $\int\int g(x,y)^2\,\mu(dx)\,\mu(dy) < \infty$. For every $f \in \mathscr{L}^2(X)$ define the function $T(f)(x) = \int g(x,y)\,f(y)\,\mu(dy)$ for $x \in X$. Prove that the function $T(f)(x)$ is well-defined for $\mu$–almost every $x \in X$, and $T(f) \in \mathscr{L}^2(X)$.

10. Let $H$ be a Hilbert space, and suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence in $H$ such that there exists an $f \in H$ with $\lim_{n\to\infty} \langle f_n, g \rangle = \langle f, g \rangle$ for all $g \in H$. Prove that $\lim_{n\to\infty} f_n = f$ if and only if $\lim_{n\to\infty} \| f_n \| = \| f \|$.

11. Find a constant $C > 0$ for which the following is true: For every $C^\infty$ function $f\colon \mathbb{R} \to \mathbb{R}$ with compact support the inequality $|f(x)|^2 \le C\,(\| f \|_2^2 + \| f' \|_2^2)$ holds for all $x \in \mathbb{R}$.

    Hint: Express both sides in terms of the Fourier–transform $\hat{f}$ of $f$, and then find an upper bound for $\frac{|f(x)|^2}{\| f \|_2^2 + \| f' \|_2^2}$ independent of $x$ and $f$.