Topics in Real Analysis
for Preparing for the Ph.D. Qualifying Examination in Analysis
August 2016 and January 2017
(prepared by Shihshu Walter Wei, Spring 2016)

The list of topics presented below is intended to be reasonably representative. The topics are fundamentally related to linear algebra, Euclidean geometry, complex analysis, topology, functional analysis, partial differential equations, harmonic analysis, geometric measure theory, differential geometry, several complex variables, geometric function theory, geometric analysis, mathematical physics, probability theory, stochastic analysis, their analogs and interconnectedness, etc. The list is not meant to be exhaustive, but to be a guide to the subject to be studied thoroughly and to lay a solid foundation in the journey of working toward the Ph.D. degree.

1. **Set Theory**: the principle of mathematical induction, inverse induction (If $P(n)$ is a proposition defined for each nature number n, then $\{P(2^n)\&[P(n) \Rightarrow P(n-1)]\} \Rightarrow (n)P(n)$), Cartesian product of two sets, unions, intersections, and complements, Boolean algebras and De Morgan's laws, partial orderings and the maximal principle.

2. **The Real Number System**: basic properties of the real number system $\mathbb{R}$ and the extended real number system $\mathbb{R} \cup \pm\infty$, open sets, closed sets, cluster points, structure theorem for open sets, infima and suprema, completeness of $\mathbb{R}$, the Cantor set and Cantor function. Notions of limit inferior and limit superior for sequences of extended real numbers $\{a_n\}$ (that is, definitions and properties of $\underline{\lim}_{n\to\infty} a_n$ and $\overline{\lim}_{n\to\infty} a_n$) and for functions $f : \mathbb{R} \to \mathbb{R}$ (that is, definitions and properties of $\underline{\lim}_{x\to a} f(x)$ and $\overline{\lim}_{x\to a} f(x)$, where $a \in \mathbb{R} \cup \pm\infty$). Upper continuous, lower continuous, continuous, and uniform continuous functions. intermediate value theorem. Fundamental theorem of calculus part I (integration $\Rightarrow$ differentiation), part II (differentiation $\Rightarrow$ integration), part III (differentiation and integration coexist, or 1-dimensional Stokes' theorem), mean value theorems for differentiation and integration. Pointwise convergence and uniform convergence of sequence of real-valued functions.

3. **Lebesgue Measure**: outer Measure, measurable sets and Lebesgue measure, measurable functions, Littlewood's three principles, existence of nonmeasurable sets.

4. **The Lebesgue Integral**: the Riemann integral. Why toward the end of 19th century mathematicians wanted to develop Lebesgue integration theory. the Lebesgue integral of a bounded function over a finite measure, the general Lebesgue Integral, Fatou's lemma, monotone convergence theorem, Lebesgue dominated convergence theorem, convergence in measure. Equality of the Riemann and Lebesgue integrals (with respect to the standard Lebesgue measure on $\mathbb{R}$) for Riemann integrable function $f : [a, b] \to \mathbb{R}$, the characterization of the Riemann integrability of of a bounded function $f : [a, b] \to \mathbb{R}$ in terms of the Lebesgue measure of its set of discontinuities.

5. **Differentiation and Integration**: *Vitali covering lemma.* Monotone functions f on $[a,b]$ are differentiable functions a.e. with its derivative f' satisfying $\int_a^b f'(x)\,dx \le f(b)-f(a)$ (compare this with FTC in Sect. **2**), functions of bounded variation, differentiation of an integral, *absolutely continuous* functions. Convex functions with geometric and physical interpretations, concave functions, scant lemma or the chordal slope lemma, regularity of convex functions (f is convex on $(a,b)\subset\mathbb{R} \Longrightarrow f$ is Lipschitz on each $[c,d]\subset(a,b) \Longrightarrow f$ is absolutely continuous on each $[c,d]\subset(a,b) \Longrightarrow f$ is differentiable a.e. in (a,b)). *characterizations of convex functions* f on $[a,b]$ when f is C^0 (f is convex $\Leftrightarrow f(\frac{1}{2}x+\frac{1}{2}y)\le\frac{1}{2}f(x)+\frac{1}{2}f(y)$ for every $x,y\in(a,b)$.), C^1 (f is convex $\Leftrightarrow f'$ is nondecreasing $\Leftrightarrow f$ is concave upward, i.e., the graph of f lies on or above all of its tangents), and C^2 (f is convex $\Leftrightarrow f''\ge 0$). maximum principle of a convex function, comparison principle of a convex function $\big($If f is a convex function on $[a,b]$, then for every $[c,d]\subset(a,b)$ and every linear function h on $[c,d]$ which satisfies $f\le h$ on the boundary $\partial[c,d]$, it holds that $f\le h$ on the interior $(c,d)\big)$, *Jensen's inequality*, convex functions in $\mathbb{R}^n$, the second derivative test for extreme values of a C^2 real-valued function of *several* real variables.

6. **The Classical Banach Spaces**: the definition of the L^p and ℓ^p spaces for both $1\le p<\infty$ and $p=\infty$, arithmetic-mean–geometric-mean inequality for any n nonnegative numbers (can be proved by using inverse induction in Sect. **1**), Young's inequality and its geometric interpretation, the Minkowski and Hölder inequalities for functions and sequences, L^p-convergence and its relation to other types of convergence (for example, L^p convergence $\Rightarrow$ convergence in measure), completeness and absolutely summable series, *Riesz-Fisher theorem*, approximations or dense subsets of $L^p[0,1]$ (simple functions, continuous functions, polynomials), *Riesz representation theorem for bounded linear functional on* $L^p[0,1]$, $1\le p<\infty$. the Banach-Sak theorem* (If a sequence $\{f_n\}$ converges to f weakly in $L^p(E)$, E is a measurable set, then there is a subsequence of arithmetic means converges strongly to f in $L^p(E)$, i.e. $\lim_{k\to\infty}\frac{f_{n_1}+\cdots+f_{n_k}}{k}=f$ strongly in $L^p(E)$).

7. **Metric Spaces**: Continuous and uniform continuous functions (between metric spaces), homeomorphisms, uniform homeomorphisms and isometries, complete metric spaces, *Borel-Lebesgue Theorem*, Cantor's diagonal method, *Arzelá-Ascoli theorem.* A metric space is compact $\Leftrightarrow$ it is both complete and totally bounded.

8. **Topological Spaces**: Continuous and uniform continuous functions (between topological spaces), homeomorphisms, subbases and bases, intermediate theorem on a connected space, uniform convergence and equicontinuity.

9. **Compact and Locally Compact Spaces**: Countable compactness, upper semicontinuity, Dini's theorem. Notions of a manifold being (a connected Hausdorff space which is) *locally Euclidean* $\big($that is, definitions of an n-dimensional differentiable manifold, a differentiable map $f: M\to N$ between differentiable manifolds, and its differential at a point $x\in M$, the inverse function theorem on manifolds : Assume that $f: M\to N$ is a differentiable map between equal dimensional differentiable manifolds, that $x\in M$, and that the differential df_x of f at x is one-to-one. Then there is an open set U containing x, such that the restriction $f_{|_U}: U\to f(U)$ is a diffeomorphism onto the open set $f(U)$ in N (no proof), and

the implicit function theorem on manifolds (no proof)). Stone-Weierstrass theorem.

10. **Banach Spaces**: Linear operators. A linear operator between normed linear spaces is continuous $\Leftrightarrow$ it is bounded. *Hahn-Banach theoerem*, the natural embedding of a normed space X into the dual X^{**} of X^*, the strong topology, the weak topology, and the weak* topology of X^*, Alaoglu theorem (no proof. This theorem is in contrast to the basic result that the closed unit ball in a Hilbert space is *not* compact with respect to the strong topology). Riesz representation theorem does not hold for $L^\infty[0,1]$. Hilbert space, parallelogram law, Pythagorean theorem, Lagrange identity, and Cauchy-Schwarz inequality, *orthogonal projection theorem and Riesz representation theorem in Hilbert space*, best approximations, least squares, Fourier coefficients, Bessel's inequality, Parseval's equation. Every bounded sequence of elements in a separable Hilbert space contains a weakly convergent subsequence (compare this with Arzelá-Ascoli theorem).

11. **General Measure and Integration**: σ-algebras of subsets of a given set X, measure on a measurable space, elementary properties of measures, definition of integral for nonnegative measurable functions on a measure space, Fatou's lemma, monotone convergence theorem, *Lebesgue dominated convergence theorem in general measure space*, signed measure, Hahn and Jordan decomposition theorems, absolute continuity of one measure with respect to another measure, *Radon-Nikodym theorem*, Radon-Nikodym derivatives, *Riesz representation theorem for* $L^p(\mu)$ with $1 \le p < \infty$ and σ-finite measure μ.

12. **Product Measures**: Definition of product measure, Fubini's theorem, Tonelli theorem, Lebesgue measure on $\mathbb{R}^n$.

13. **Hausdorff Measures**: Borel set and Borel measure, definition of Hausdorff dimension. The Hausdorff dimension of the Cantor set is $\frac{\log 2}{\log 3}$.

14. **The Daniel Integral** (Bonus Topic): basic properties of vector lattice, positive linear functional and a Daniel functional or a Daniel integral.

* The Banach-Sak theorem plays a crucial role in proving that a continuous convex functional on a closed, bounded, convex subset of $L^p(E)$ takes a minimum value. This arguments Weierstrass theorem (Every continuous real-valued function on a closed, bounded interval takes a minimum value), and proves the existence of a minimizer for a *specific* functional. It is known that a minimizer does not exist for a *general* functional.

References

1. Real Analysis (3rd edition), by H.L.Royden
2. Real Analysis (4th edition), by H.L.Royden and P.M. Fitzpatrick
3. Real Analysis (2nd edition), by G.B. Folland
4. Analysis (2nd edition), by E.H. Lieb and M. Loss
5. Real and Complex Analysis (3rd edition), by W. Rudin
6. An Introduction to Measure Theory, by Terrence Tao