**Real Analysis**

**Qualifying Examination**

**Spring 2017**

**NAME:** **I.D. # :**

**Complete five (5) of the problems below. If you attempt more than 5 questions, then please clearly indicate which 5 should be graded on this sheet.**

1a. Consider the following statement: Let $(X, \mathcal{B}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence in $L^1(\mu)$ that converges uniformly on $X$ to a function $f \in L^1(\mu)$ ; then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

If the above statement is true prove it. If the above statement is false, $(i)$ show by example and $(ii)$ add a hypothesis to the above statement that the results is a true statement, and give a proof that your modified statement is indeed true.

Let $(X, \mathcal{B}, \mu)$ be a measure space.
1b. State the following theorems.
$(i)$ Fatou's Lemma
$(ii)$ Monotone Convergence Theroem
$(iii)$ Dominated Convergence Thereom

1c. Prove that $(ii)$ implies $(i)$.

2a. State Vitali covering lemma.

2b. Let $f$ be an increasing real-valued function on $[a, b]$ , and

$$E_{u,v} = \{x : D^+ f(x) > u > v > D_- f(x)\},$$

where $u$ and $v$ are rational numbers,

$$D^+ f(x) = \overline{\lim_{h\to 0+}} \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad D_- f(x) = \underline{\lim_{h\to 0+}} \frac{f(x) - f(x-h)}{h}$$

Prove that the outer measure $m^*(E_{u,v}) = 0$ .

3a. State Ascoli-Arzelá Theorem $\big($on a metric space $(X, d)\big)$

3b. Let $\Delta$ be the unit disk on the complex plane ( i.e. $\{z : |z| < 1\}$) . Let $\{f_n\}_1^\infty$ be the sequence of analytic functions on $\Delta$ such that $|f_n(z)| \le M$ independent of $n$ and $z \in \Delta$ . Prove that there is a subsequence $f_{n_k}$ which converges uniformly to every compact subset of $\Delta$ . (Hint: Use the Cauchy formula $f_n(z) = \frac{1}{2\pi i}\int_C \frac{f_n(w)}{w-z}\, dw$ and Ascoli-Arzelá theorem)

4a. Let $f$ be a real-valued twice differentiable convex function on an open interval $(a, b)$ . How does a relevant algebraic inequality meet a differential inequality and how does a differential inequality meet an algebraic inequality? Justify your answers.

4b. Use the convexity to prove Jensen's inequality.

4c. Use #4b to prove an arithmetic mean is no less than a geometric mean for $a_1, \dots, a_n > 0$ .

5a. Define an $n$-dimensional differentiable manifold.

5b. State the implicit function theorem on a differentiable manifold.

6a. State Orthogonal Projection Theorem in Hilbert Space.

6b. Let $M$ be a subspace of a a Hilbert space $V$. Let $x$ be in $V$ Prove that if $y$ is in the subspace $M$, then $(x - y) \perp M$ if and only if $y$ is the unique point in $M$ closest to $x$, that is, $y$ **is the "best approximation" to** $x$ **in** $M$

6c. State Riesz Representation Theorem in Hilbert Space

6d. Let $f$ be a linear functional on a Hilbert space $V$, $M = \{x \in V | f(x) = 0\}$, and $M \neq V$. Let $x \in V \backslash M$ and $x^\perp$ be its component orthogonal to $M$. Find an explicitly constant $c$ so that

$$x - cx^\perp \quad \text{is orthogonal to} \quad x^\perp .$$

7a. Let $f$ denote a real-valued, continuous and strictly increasing function on $[0, c]$ with $c > 0$ and $f(0) = 0$. Let $f^{-1}$ denote the inverse function of $f$. Then, for all $a \in [0, c]$ and $b \in [0, f(c)]$

$$(*) \qquad ab \leq \int_0^a f(x)\, dx + \int_0^b f^{-1}(x)\, dx$$

with equality if and only if $b = f(a)$.

7b. Find an appropriate $f$ in # 7a so that Cauchy inequality is a special case of $(*)$ in #7a.

7c. Find an appropriate $f$ in # 7a so that Young's inequality is a special case of $(*)$ in #7a.

Recall that a real-valued function $f$ on $[0, 1]$ is said to be Hölder continuous of order $\alpha$ if there exists a constant $C$ such that

$$|f(x) - f(y)| \leq C|x - y|^\alpha$$

for every $x, y \in [0, 1]$. Define

$$\|f\|_\alpha = \max \left| f(x) + \sup \frac{|f(x) - f(y)|}{|x - y|^\alpha} \right|$$

8. Let $C^{0,\alpha}[0, 1]$ be the set of Hölder continuous function $f$ defined on $[0, 1]$ of order $\alpha$. Prove that $C^{0,\alpha}[0, 1]$ under $\|f\|_\alpha$ is a Banach space.

9a. State Hahn-Banach Theorem

9b. Let $X$ be a complex vector space, $S$ a linear subspace, $p$ a real-valued function on $X$ such that $p(x + y) \leq p(x) + p(y)$, and $p(\alpha x) = |\alpha| p(x)$. Let $f$ be a complex linear functional on $S$ such that $|f(s)| \leq p(s)$ for all $s$ in $S$. Use #9a to prove that there is a linear functional $F$ defined on $X$ such that $F(s) = f(s)$ for $s \in S$ and $|F(x)| \leq p(x)$ for all $x$ in $X$.

10a. State Radon-Nikodym Theorem

10b. Let $\mu$ and $\nu$ be $\sigma$-finite. Show that if $\nu << \mu$ and $\mu << \nu$, then their Radon-Nikodym derivatives satisfy

$$\left[\frac{d\nu}{d\mu}\right] = \left[\frac{d\mu}{d\nu}\right]^{-1}$$

where $\nu << \mu$ denote $\nu$ is absolutely continuous with respect to $\mu$.

11a. State the Fubini theorem.

11b. Let $X = Y = [0,1]$, $\mu =$ counting measure on $[0,1]$, $\lambda =$ Lebegue measure on $Y$. Let

$$f(x) = \begin{cases} 1 & \text{if} \quad x = y \\ 0 & \text{otherwise} \end{cases}$$

Does the Fubini theorem hold in this case? Justify your answer?

12a. Let $f$ be a continuously differentiable on an interval containing $[a,b]$. Prove that

$$(**) \qquad \int_a^b f'(x)\,dx = f(b) - f(a)\,.$$

12b. Let $f$ be an increasing, real-valued, differentiable a.e. on $[a,b]$ and the derivative $f'$ is measurable. Prove or disprove that the same conclusion $(**)$ holds in #12a for $f$?

12c. Prove that if $f$ is absolutely continuous on each $[c,d] \subset (a,b) \subset \mathbb{R}$, $f$ is differentiable a.e. in $(a,b)$.

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