Ph.D. Qualifying Exam in Analysis
August 14, 2017

There are 9 questions on this three-hour exam. Answer as many of them as you can.

1. Let $E \subseteq R$ and suppose that for every $\epsilon > 0$ there exist open subsets U and V of $\mathbb{R}$ such that $U \subseteq E \subseteq V$ and $m(V - U) < \epsilon$, where m is Lebesgue measure. Prove that E is measurable.

2. Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of integrable functions on a measure space $(X, \mathcal{A}, \mu)$, and f is an integrable function on X such that $\sum_{n=1}^{\infty} \int_X |f - f_n| \, d\mu < \infty$. Show that f_n converges to f pointwise almost everywhere on X.

3. Suppose $\{f_n\}_{n\in\mathbb{N}}$, f, and g are integrable functions on a measure space $(X, \mathcal{A}, \mu)$, such that f_n converges to f in measure, and $|f_n| \leq g$ on X for all $n \in \mathbb{N}$. Prove that $\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0$. (Hint: to prove that a sequence a_n converges to zero, it is enough to show that every subsequence of a_n has, in turn, a subsubsequence that converges to zero.)

4. Suppose $\mu(x)$ is a finite Borel measure on $\mathbb{R}$ such that $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$, and define $g_\mu(x) := \mu([0, x])$ for $x \in \mathbb{R}$. Show that g_μ is continuous on $\mathbb{R}$.

5. Suppose g is a nonnegative measurable function on $\mathbb{R}$, and there exists a constant $C > 0$ such that for all $f \in L^2$, we have $gf \in L^2$ with $\|gf\|_{L^2} \leq C\|f\|_2$. Show that $g \in L^\infty$ with $\|g\|_\infty \leq C$. (Hint: consider $f(x) = \chi_{[x_0, x_0+h]}$ for $x_0 \in \mathbb{R}$ and $h > 0$.)

6. Suppose $Q(x, y) : \mathbb{R}\times\mathbb{R} \to [0, \infty)$ is measurable, and $\int_{\mathbb{R}} Q(x, y) \, dy \leq a$ for all $x \in \mathbb{R}$ and $\int_{\mathbb{R}} Q(x, y) \, dx \leq b$ for all $y \in \mathbb{R}$. For $f(x)$ nonnegative and measurable, define $Tf(x) := \int_{\mathbb{R}} Q(x, y)f(y) \, dy$. Show that $\|Tf\|_{L^2} \leq \sqrt{ab}\|f\|_{L^2}$.

7. Let $f(x)$ be the Cantor-Lebesgue function on $[0, 1]$, and define

$$g(x) = \begin{cases} 0 & -\infty < x \leq 0 \\ f(x) & 0 \leq x \leq 1 \\ x & x \geq 1. \end{cases}$$

   Let μ_g be the Lebesgue-Stieltjes measure on $\mathbb{R}$ corresponding to g (so $\mu_g((a, b]) = g(b) - g(a)$) for all $a, b \in \mathbb{R}$). Find the Lebesgue decomposition of μ_g with respect to Lebesgue measure m. That is, define two measures μ and ν such that $\mu_g = \mu + \nu$, $\mu << m$, and $\nu \perp m$.

8. Suppose $f : [0, \infty) \to \mathbb{R}$ is of bounded variation on $[0, \infty)$. Show that $\lim_{x\to\infty} f(x)$ exists.

9. Suppose f is differentiable with continuous derivative on $[0, 2\pi]$, and $f(0) = f(2\pi)$. For $n \in \mathbb{Z}$, define

$$a_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x)e^{-inx} \, dx.$$

   (a) Show that for $n \in \mathbb{Z}$,

$$\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f'(x)e^{-inx} \, dx = ina_n.$$

   (b) Conclude from (a) that $\sum_{n=-\infty}^{\infty} n^2|a_n|^2 < \infty$.

   (c) Conclude from (b) that $\sum_{n=-\infty}^{\infty} |a_n| < \infty$.