Spring 2018

TOPOLOGY QUALIFY EXAM

Instructions: You have three hours to finish the exam. There are three parts to the exam. For the proofs, provide justifications and make your arguments clear, giving an appropriate level of detail.

I

Definitions and statements of theorems (solve ALL problems)

1. State the definition of a partition of unity.
2. Define what it means for a topological space to be normal.
3. What does it mean for two paths in a topological space X to be path homotopic?
4. State Tietze's extension theorem.
5. State the Borsuk-Ulam theorem for S^2.

II

Solve FOUR of these problems.

1. Consider $\mathbb{R}$ with the topology given by a basis with sets of the form (a,b) and $[a,b)$. Determine whether the set $[0,1]$ is compact.
2. Show that S^2 is connected.
3. Is the one point compactification of a manifold a manifold? Hint: consider a cylinder.
4. Show the product of two connected spaces is connected.
5. Prove that a surjective map which is either open or closed is a quotient map. Give an example of a quotient map which is neither open nor closed.

III

Solve FOUR of these problems.

1. Consider the action of $\mathbb{Z}_2$ on S^2 where the non-identity element sends a point p to the antipodal point $-p$. Prove the action is properly discontinuous.
2. Consider a map $f : X \to B$. Assume $\pi_1(X) = \mathbb{Z}_3$ and $\pi_1(B) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Can you say for which covering maps $p : E \to B$ the map f has a lift?
3. Given an example of a 2-dimensional cell complex which is not contractible. Briefly explain what it is not contractible.
4. Let A be a deformation retract of X. Prove the inclusion map induces an isomorphism of fundamental groups.
5. Consider a disk D^2 with boundary S^1. Consider the group $\mathbb{Z}_n$ acting on the boundary where $i \in \mathbb{Z}_n$ acts via rotation by $\frac{i\,2\pi}{n}$. Let X be the quotient of D^2 obtained by identifying points on S^1 in the same orbit of $\mathbb{Z}_n$. (Note, points on the interior of D^2 are not identified in the quotient.) Compute the fundamental group of X.