Algebra Qualifying Exam August 15, 2018

Name:

i) Let G be a group of size 75. Let P be the Sylow 5-subgroup of G and let Q be a subgroup of G of order 15. Show that $H = P \cap Q$ is a non-trivial cyclic subgroup of G.

ii) Let $G = \mathrm{SL}_2(\mathbb{R}) := \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}, ad - bc = 1\}$ and let $\mathbb{H} := \{z = x + iy \in \mathbb{C} : y > 0\}$ be the complex upper half plane. Given $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $z \in \mathbb{H}$, define

$$g\langle z \rangle := \frac{az + b}{cz + d}.$$

a) For $g \in G$ and $z \in \mathbb{H}$, show that $g\langle z \rangle \in \mathbb{H}$.

b) For $g_1, g_2 \in G$ and $z \in \mathbb{H}$, show that $g_1\langle g_2\langle z \rangle\rangle = (g_1 g_2)\langle z \rangle$.

c) Show that

$$g\langle i \rangle = i \Leftrightarrow g = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \text{ with } \theta \in [0, 2\pi].$$

iii) Define $R := \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a \equiv b \pmod 3\}$. (Recall that $a \equiv b \pmod 3$ means $3|(a - b)$.)

a) Show that R is a subring of $\mathbb{Z} \times \mathbb{Z}$.

b) Consider the ring homomorphism $f : \mathbb{Z}[x] \to R$ defined by $f(1) = (1, 1), f(x) = (3, 0)$ and extended additively and multiplicatively to $\mathbb{Z}[x]$. Show that f is surjective and that kernel of f is the principal ideal $(x^2 - 3x)$.

iv) a) Show that every element of $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ can be written in the form $x \otimes 1$ for some $x \in \mathbb{Q}$. (Recall that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ includes sums of elements of the form $a \otimes b$.)

b) Show that the map $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ given by $a \otimes b \mapsto ab$ is an isomorphism of additive abelian groups.

v) Let $\alpha = \sqrt{4 + \sqrt{7}}$.

a) Find the minimal polynomial f of α over $\mathbb{Q}$. (Make sure that you show that f is irreducible)

b) Let K be the splitting field of f over $\mathbb{Q}$. Show that the degree of K over $\mathbb{Q}$ is 8.

c) Let $\mathrm{Gal}(K/\mathbb{Q})$ be the Galois group of K over $\mathbb{Q}$. Give a concrete example of an element $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ of order 4 and an element $\tau \in \mathrm{Gal}(K/\mathbb{Q})$ of order 2 such that $\tau \neq \sigma^2$.

d) Show that the group generated by σ and τ is the dihedral group D_4 of order 8. Hence, conclude that $\mathrm{Gal}(K/\mathbb{Q}) = \langle \sigma, \tau \rangle$, the group generated by σ and τ. (Recall, the dihedral group D_n of size $2n$ is given in terms of generators by $\langle r, s | r^n = s^2 = 1, srs = r^{-1} \rangle = \langle r, s | r^n = s^2 = (sr)^2 = 1 \rangle$.)

vi) Let $\alpha \in \mathbb{C}$ such that $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ is finite and odd. Show that $\mathbb{Q}(\alpha^2) = \mathbb{Q}(\alpha)$.