Algebra Qualifying Exam January 7, 2019

Name:

i) a) Write $\sigma = (456)(23)(12)(678)$ as a product of disjoint cycles and find the order of σ.

b) Let $n > 1$ be an odd integer. Show that S_n has at least $n-1$ elements of order $2(n-2)$.

ii) Let G be a group. Recall that the *commutator* of two elements $g, h \in G$ is defined as $[g,h] := g^{-1}h^{-1}gh$. Also recall that the commutator subgroup of G is its subgroup $[G,G]$ generated by commutators $[g,h]$, as g, h vary over all the elements of G.

a) Show that $[G,G]$ is a normal subgroup of G.

b) Show that $G/[G,G]$ is abelian.

c) Show that if H is a subgroup of G containing $[G,G]$, then H is normal.

iii) Let A be a commutative ring. An element $a \in A$ is called nilpotent if there is a positive integer n such that $a^n = 0$. The *nilradical* N of A is the subset of A consisting of all the nilpotent elements of A.

a) Suppose a, b are nilpotent elements of A. Show that $-a, a+b, ab$ are also nilpotent. Conclude that the nilradical N is a subring of A.

b) Show that the nilradical N is an ideal of A.

c) Show that the nilradical N is contained in every prime ideal of A.

iv) Let F be a field and let V be a finite dimensional F-vector space. Let V^* denote the dual of V, i.e. V^* is the vector space of all linear functionals from V to F. Let $\mathrm{End}_F(V)$ be the F-vector space of all linear transformations $V \to V$.

a) For $v \in V$ and $f \in V^*$, let $T_{v,f} \in \mathrm{End}_F(V)$ be defined by $T_{v,f}(w) := f(w)v$ for all $w \in V$. Show that the map $\psi : V \times V^* \to \mathrm{End}_F(V)$ given by $\psi(v,f) := T_{v,f}$ is a F-bilinear map.

b) Show that ψ induces an **isomorphism** $\tilde{\psi} : V \otimes_F V^* \to \mathrm{End}_F(V)$.

v) a) Show that $\alpha = \sqrt{5} + \sqrt{7}$ is algebraic over $\mathbb{Q}$ by explicitly finding a polynomial $f(x) \in \mathbb{Q}[x]$ of degree 4 having α as a root.

b) Show that the $f(x)$ from part a) is irreducible.

c) Show that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois and determine its Galois group. Justify your answer.

d) Illustrate the Fundamental Theorem of Galois Theory by drawing the lattice of intermediate fields and the corresponding subgroups. You don't have to prove your answer.

vi) Let $p \geq 3$ be a prime number. Let $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$ be the finite fields of size p and p^2 respectively. Show that the map $\phi : \mathbb{F}_{p^2} \to \mathbb{F}_{p^2}$ defined by $\phi(x) := x^p$ is an automorphism of $\mathbb{F}_{p^2}$ that fixes $\mathbb{F}_p$.